Popular Astronomy.

Vol. XIII No. 6. JUNE AND JULY, 1905. Whole No. 126

ELLIPTIC MOTION.

ASAPH HALL.

FOR POPULAR ASTRONOMY.

The difference from unity of the ratio of the radius vector and the mean distance, and the equation of center, are two important quantities in the theory of elliptic motion. The equation of center is the more difficult to determine. If this equation is wanted in explicit terms of the mean anomaly and the excentricity perhaps the easiest method is to employ the equation

$$dv = \sqrt{1-e^2}\cdot\frac{a^2}{r^2}\,ndt.$$

The ratio $\frac{a^2}{r^2}$ can be found by the theorem of Lagrange in terms of the mean anomaly, nt, and the excentricity, and then an integration gives $v - nt$, or the equation of center.

In his "*Mecanique*" Poisson has indicated a method of obtaining these quantities by means of definite integrals.

Assume

$$r = A_0 + A_1 \cos nt + A_2 \cos 2nt + \ldots\ldots + A_i \cos int + \ldots\ldots$$

$$v - nt = \quad B_1 \sin nt + B_2 \sin 2nt + \ldots\ldots + B_i \sin int + \ldots\ldots$$

Multiply the series for r by $\cos int\,.\,dnt$; and that for $v - nt$ by $\sin nt\,.\,dnt$: When the integration extends from o to π, and i and i' are positive whole numbers, and different, all the integrals vanish except,

$$A_i = \frac{2}{\pi}\cdot\int_0^{\pi} r \cos int.\,dnt$$

and

$$B_i = \frac{2}{\pi}\cdot\int_0^{\pi} (v - nt) \sin int.\,dnt$$

In the case of $i = i'$, the value of the integral is $\frac{\pi}{2}$, and for $i = 0$, we have an exception: the first integral is π, and the second is zero. Thus

$$A_0 = \frac{1}{\pi} \cdot \int_0^{\pi} r \, . \, dnt$$

If u be the excentric anomaly then

$$r = a\,(1 - e\cos u)$$
$$dnt = (1 - e\cos u)\,du$$

and

$$A_0 = \frac{a}{\pi} \cdot \int_0^{\pi} (1 - e\cos u)^2 \, . \, du = a\,(1 + \frac{1}{2}\,e^2).$$

This is the only coefficient that can be found in finite terms. Since v, u, and nt, have the values 0 and π at the same time, if we integrate by parts we have

$$B_i = \frac{2}{i\pi} \int_0^{\pi} \cos int \, . \, d\,(v - nt)$$

But

$$nt = u - e\sin u; \qquad dv = \frac{\sqrt{1-e^2} \, . \, du}{1 - e\cos u},$$

and changing the variable the coefficients become,

$$A_i = \frac{2a}{\pi} \cdot \int_0^{\pi} (1 - e\cos u)^2 \, . \cos\,(iu - ie\sin u) \, . \, du$$

$$B_i = \frac{2}{i\pi} \cdot \int_0^{\pi} \left\{ \sqrt{1-e^2} - (1 - \cos u)^2 \right\} \cdot \frac{\cos\,(iu - ie\sin u)}{1 - e\cos u} \, . \, du$$

But $\cos\,(iu - ie\sin u) = \cos iu\cos ie\sin u + \sin iu\sin ie\sin u$, or putting the series for the small terms

$$\cos\,(iu - ie\sin u) = \left(1 - \frac{i^2e^2\sin^2 u}{\lfloor 2} + \frac{i^4e^4\sin^4 u}{\lfloor 4} - \frac{i^6e^6\sin^6 u}{\lfloor 6} + \ldots\ldots\right)\cos iu$$
$$+ \left(\frac{ie\sin u}{1} - \frac{i^3e^3\sin^3 u}{\lfloor 3} + \frac{i^5e^5\sin^5 u}{\lfloor 5} - \ldots\ldots\ldots\ldots\right)\sin iu$$

The differential of $\sin\,(iu - ie\sin u) =$
$$i\,.\,\cos\,(iu - ie\sin u)(1 - e\cos u)\,.\,du$$

and hence

$$A_i = \frac{2a}{i\pi} \cdot \int_0^{\pi} (1 - e\cos u)\,.\,d.\sin\,(iu - ie\sin u)$$

An integration by parts gives

$$(1 - e\cos u)\,\sin iu(-ie\sin u) - e \cdot \int_0^{\pi} \sin\,(iu - ie\sin u)\sin u\,du$$

The first term is zero at the limits, and we have finally,

$$A_i = -\frac{2ae}{i\pi} \cdot \int_0^{\pi} \sin\,(iu - ie\sin u)\sin u\,du \qquad (a)$$

By expansion

$$\sin\,(iu - ie\sin u)\sin u\,du$$
$$= +\sin iu\left(\sin u - \frac{i^2e^2\sin^2 u}{\lfloor 2} + \frac{i^4e^4\sin^4 u}{\lfloor 4} - \frac{i^6e^6\sin^6 u}{\lfloor 6} + \ldots\ldots\ldots\right).du$$
$$-\cos iu\left(\frac{ie\sin^2 u}{1} - \frac{i^3e^3\sin^4 u}{\lfloor 3} + \frac{i^5e^5\sin^6 u}{\lfloor 5} - \frac{i^7e^7\sin^8 u}{\lfloor 7} + \ldots\ldots\right).du$$

The odd powers of $\sin u$ can be resolved into sines of odd multiples of u; and the even powers of $\sin u$ into cosines of even multiples of u. As these are multiplied by $\sin iu$, and $\cos iu$, the forms to integrate will be $\sin iu \sin i'u$, and $\cos iu \cos i'u$. If i and i' are different the integral is zero; if $i = i'$, the result is $\frac{\pi}{2}$, which will cancel the factor $\frac{2}{\pi}$ outside the integral. Make $i = 1$, and the first line gives,

$$\left(1 - \frac{e^2}{\lfloor 2} \cdot \frac{3}{4} + \frac{e^4}{\lfloor 4} \cdot \frac{5}{8} - \frac{e^6}{\lfloor 6} \cdot \frac{35}{64} + \frac{e^8}{\lfloor 8} \cdot \frac{63}{128} - \ldots\right) \frac{\pi}{2} \times \left(-\frac{ae}{i}\right)$$

The second line gives zero. Omit the factor a and we have

$$A_1 = -e + \frac{3}{8}e^3 - \frac{5}{192}e^5 + \frac{7}{9216}e^7 - \frac{1}{81920}e^9 + \ldots\ldots$$

When $i = 2$, the first line gives nothing, and so on alternately: Hence

$$A_2 = -\frac{e^2}{2} + \frac{e^4}{3} - \frac{e^6}{16} + \frac{e^8}{180} - \ldots\ldots$$

$$A_3 = -\frac{3}{8}e^3 + \frac{45}{128}e^5 - \frac{567}{5120}e^7 + \frac{729}{40960}e^9 - \ldots\ldots$$

$$A_4 = -\frac{1}{3}e^4 + \frac{2}{5}e^6 - \frac{8}{45}e^8 + \ldots\ldots$$

$$A_5 = -\frac{125}{384}e^5 + \frac{4375}{9216}e^7 - \frac{15625}{57344}e^9 + \ldots\ldots$$

$$A_6 = -\frac{27}{80}e^6 + \frac{81}{140}e^8 - \ldots\ldots$$

$$A_7 = -\frac{16807}{46080}e^7 + \frac{117649}{163840}e^9 - \ldots\ldots$$

$$A_8 = -\frac{128}{315}e^8 + \ldots\ldots$$

$$A_9 = -\frac{531441}{1146880}e^9 + \ldots\ldots$$

These are the coefficients A_1 to e^9 inclusive.

Multiplying the factors of B_i

$$B_i = \frac{2}{i\pi} \cdot \int_0^{\pi} \cdot \left\{ \frac{\sqrt{1-e^2} \cdot \cos(iu - ie\sin u)\, du}{1 - e\cos u} - \frac{\cos(iu - ie\sin u)(1 - e\cos u)\, du}{1} \right\}$$

The last term is the differential of $\frac{1}{i} \cdot \sin(iu - ie\sin u)$, which is zero at the limits, and we have

$$B_i = \frac{2\sqrt{1-e^2}}{i\pi} \cdot \int_0^{\pi} \cdot \frac{\cos(iu - ie\sin u)\, du}{1 - e\cos u} \qquad (b)$$

Expanding the quantity under the sign of integration,

$$\cos(iu - ie\sin u)(1 + e\cos u + e^2\cos^2 u + e^3\cos^3 u + e^4\cos^4 u + \ldots\ldots)\, du$$

or arranging the full expansion in lines,

$$+\left(1-\frac{i^2e^2\sin^2 u}{\lfloor 2}+\frac{i^4e^4\sin^4 u}{\lfloor 4}-\frac{i^6e^6\sin^6 u}{\lfloor 6}+\ldots\ldots\ldots\right)\cos iu\,.\,du \qquad (1)$$

$$+\left(ie\sin u-\frac{i^3e^3\sin^3 u}{\lfloor 3}+\frac{i^5e^5\sin^5 u}{\lfloor 5}-\ldots\ldots\ldots\ldots\right)\sin iu\,.\,du \qquad (2)$$

$$+\left(e-\frac{i^2e^3\sin^2 u}{\lfloor 2}+\frac{i^4e^5\sin^4 u}{\lfloor 4}-\frac{i^6e^7\sin^6 u}{\lfloor 6}+\ldots\ldots\ldots\right)\cos iu\cos u\,.\,du \qquad (3)$$

$$+\left(ie^2\sin u-\frac{i^3e^4\sin^3 u}{\lfloor 3}+\frac{i^5e^6\sin^5 u}{\lfloor 5}-\ldots\ldots\ldots\ldots\right)\sin iu\cos u\,.\,du \qquad (4)$$

$$+\left(e^2-\frac{i^2e^4\sin^2 u}{\lfloor 2}+\frac{i^4e^6\sin^4 u}{\lfloor 4}-\frac{i^6e^8\sin^6 u}{\lfloor 6}+\ldots\ldots\ldots\right)\cos iu\cos^2 u\,.\,du \qquad (5)$$

$$+\left(ie^3\sin u-\frac{i^3e^5\sin^3 u}{\lfloor 3}+\frac{i^5e^7\sin^5 u}{\lfloor 5}-\ldots\ldots\ldots\ldots\right)\sin iu\cos^2 u\,.\,du \qquad (6)$$

$$+\left(e^3-\frac{i^2e^5\sin^2 u}{\lfloor 2}+\frac{i^4e^7\sin^4 u}{\lfloor 4}-\frac{i^6e^9\sin^6 u}{\lfloor 6}+\ldots\ldots\ldots\right)\cos iu\cos^3 u\,.\,du \qquad (7)$$

$$+\left(ie^4\sin u-\frac{i^3e^6\sin^3 u}{\lfloor 3}+\frac{i^5e^8\sin^5 u}{\lfloor 5}-\ldots\ldots\ldots\ldots\right)\sin iu\cos^3 u\,.\,du \qquad (8)$$

etc. etc. etc.

The first two lines can be integrated like the A_i coefficients: the third line will give terms of the form $\sin^n u\cos iu\,.\,d\sin u$. Integrate by parts and the result is

$$\frac{1}{n+1}\,.\,\sin^{n+1}u\cos iu+i\int\frac{1}{n+1}\sin^{n+1}u\sin iu\,.\,du$$

The first term is zero at the limits, and the integral becomes,

$$B'_i=\frac{2}{\pi}\,.\,\frac{\sqrt{1-e^2}}{n+1}\,.\int_0^{\pi}\sin^{n+1}u\sin iu\,.\,du$$

For line (4) the form is $\sin^n u\sin iu\,.\,d\sin u$ which comes to

$$\frac{1}{n+1}\,.\,\sin^{n+1}u\sin iu-\frac{i}{n+1}\int\sin^{n+1}u\cos iu\,.\,du$$

and the integral is

$$B_i''=-\frac{2}{\pi}\,.\,\frac{\sqrt{1-e^2}}{n+1}\,.\int_0^{\pi}\sin^{n+1}u\cos in\,.\,du$$

If in lines (5) and (6) we change $\cos^2 u$ to $1-\sin^2 u$ we return to (1) and (2); and in (7) and (8) $\cos^3 u=(1-\sin^2 u)\cos u$, and we return to (3) and (4). These results will alternate through all the lines, and the three preceding forms of B_i suffice for the integrations. The coefficients of A_i consist of odd powers of e when i is odd, and we may infer that this will be the case with the B_i coefficients. It will be best to expand the forms fully, and

then make i, 1, 2, 3, 4, 5, 6, 7, 8, 9, successively. The integrations are very simple and we must multiply the result for each value of i by

$$\sqrt{1-e^2} = 1 - \frac{1}{2}e^2 - \frac{1}{8}e^4 - \frac{1}{16}e^6 - \frac{5}{128}e^8 - \ldots\ldots$$

I give some of the results for the smaller values of i, and which are the more difficult to compute.

$i = 1.$

Line (2).	$+e-\frac{e^3}{8}$	$+\frac{e^5}{192}$	$-\frac{e^7}{9216}$	$+\frac{e^9}{737280}$
" (3).	$+e-\frac{e^3}{8}$	$+\frac{e^5}{192}$	$-\frac{e^7}{9216}$	$+\frac{e^9}{737280}$
" (6).	$+\frac{e^3}{4}$	$-\frac{e^5}{48}$	$+\frac{e^7}{1536}$	$-\frac{e^9}{92160}$
" (7).	$+\frac{3e^3}{4}$	$-\frac{e^5}{16}$	$+\frac{3e^7}{1536}$	$-\frac{3e^9}{92160}$
" (10).		$+\frac{1}{8}e^5$	$-\frac{1}{128}e^7$	$+\frac{1}{5120}e^9$
" (11).		$+\frac{5}{8}e^5$	$-\frac{5}{128}e^7$	$+\frac{1}{1024}e^9$
" (14).			$+\frac{5}{64}e^7$	$-\frac{1}{256}e^9$
" (15).			$+\frac{35}{64}e^7$	$-\frac{7}{256}e^9$
" (18).				$+\frac{7}{128}e^9$
" (19).				$+\frac{63}{128}e^9$
Sum =	$+2e+\frac{3}{4}e^3$	$+\frac{65}{96}e^5+$	$\frac{2675}{4608}e^7$	$+\frac{190497}{368640}e^9$
Mult. by	$+1-\frac{1}{2}e^2$	$-\frac{1}{8}e^4-$	$\frac{1}{16}e^6$	$-\frac{5}{128}e^8-\ldots\ldots$
	$+2e+\frac{3}{4}e^3$	$+\frac{65}{96}e^5+$	$\frac{2675}{4608}e^7$	$+\frac{190497}{368640}e^9$
	$-e^3$	$-\frac{3}{8}e^5-$	$\frac{65}{192}e^7$	$-\frac{2675}{9216}e^9$
		$-\frac{1}{4}e^5-$	$\frac{3}{32}e^7$	$-\frac{65}{768}e^9$
			$-\frac{1}{8}e^7-$	$\frac{3}{64}e^9$
				$-\frac{5}{64}e^9$
$\therefore B_1 =$	$+2e-\frac{1}{4}e^3$	$+\frac{5}{96}e^5+$	$\frac{107}{4608}e^7$	$+\frac{6217}{368640}e^9$

$i = 2.$

Line	(1).	$+\frac{e^2}{2}$	$-\frac{e^4}{6}$	$+\frac{e^6}{48}$	$-\frac{e^8}{720}$
"	(4).	$+\frac{e^2}{2}$	$-\frac{e^4}{6}$	$+\frac{e^6}{48}$	$-\frac{e^8}{720}$
"	(5).	$+\frac{e^2}{4}$	-0	$-\frac{1}{96}e^6$	$+\frac{e^8}{720}$
"	(8).		$+\frac{e^4}{4}$	$-\frac{1}{16}e^6$	$+\frac{3}{480}e^8$
"	(9).		$+\frac{e^4}{4}$	$-\frac{1}{32}e^6$	-0
"	(12).			$+\frac{5}{32}e^6$	$-\frac{1}{32}e^8$
"	(13).			$+\frac{15}{64}e^6$	$-\frac{1}{32}e^8$
"	(16).				$+\frac{7}{64}e^8$
"	(17).				$+\frac{7}{32}e^8$
	Sum $=$	$+\frac{5}{4}e^2$	$+\frac{1}{6}e^4$	$+\frac{21}{64}e^6$	$+\frac{779}{2880}e^8$
	Mult. by	$+1$	$-\frac{1}{2}e^2$	$-\frac{1}{8}e^4$	$-\frac{1}{16}e^6$
		$+\frac{5}{4}e^2$	$+\frac{1}{6}e^4$	$+\frac{21}{64}e^6$	$+\frac{779}{2880}e^8$
			$-\frac{5}{8}e^4$	$-\frac{1}{12}e^6$	$-\frac{21}{128}e^8$
				$-\frac{5}{32}e^6$	$-\frac{1}{48}e^8$
					$-\frac{5}{64}e^8$
	$\therefore B_2 =$	$+\frac{5}{4}e^2$	$-\frac{11}{24}e^4$	$+\frac{17}{192}e^6$	$+\frac{43}{5760}e^8$

$i = 3.$

Line	(2).	$+\frac{3}{8}e^3$	$-\frac{27}{128}e^5$	$+\frac{243}{5120}e^7$	$-\frac{243}{40960}e^9$
"	(3).	$+\frac{3}{8}e^3$	$-\frac{27}{128}e^5$	$+\frac{243}{5120}e^7$	$-\frac{243}{40960}e^9$
"	(6).	$+\frac{1}{4}e^3$	$-\frac{3}{32}e^5$	$+\frac{27}{2560}e^7$	-0
"	(7).	$+\frac{1}{12}e^3$	$+\frac{3}{32}e^5$	$-\frac{27}{512}e^7$	$+\frac{54}{5120}e^9$
"	(10).		$+\frac{3}{16}e^5$	$-\frac{9}{128}e^7$	$+\frac{27}{2560}e^9$

Line (11). $+\frac{5}{48}e^5+\frac{3}{128}e^7-\frac{9}{512}e^9$

" (14). $+\frac{9}{64}e^7-\frac{3}{64}e^9$

" (15). $+\frac{7}{64}e^7-0$

" (18). $+\frac{7}{64}e^9$

" (19). $+\frac{7}{64}e^9$

Sum $=+\frac{13}{12}e^3-\frac{25}{192}e^5+\frac{131}{512}e^7+\frac{3349}{20480}e^9$

Mult. by $+1-\frac{1}{2}e^2-\frac{1}{8}e^4-\frac{1}{16}e^6$

$+\frac{13}{12}e^3-\frac{25}{192}e^5+\frac{131}{512}e^7+\frac{3349}{20480}e^9$

$-\frac{13}{24}e^5+\frac{25}{384}e^7-\frac{131}{1024}e^9$

$-\frac{13}{96}e^7+\frac{25}{1536}e^9$

$-\frac{13}{192}e^9$

$\therefore B_3=+\frac{13}{12}e^3-\frac{43}{64}e^5+\frac{95}{512}e^7-\frac{973}{61440}e^9$

$i=4.$

Line (1). $+\frac{e^4}{3}-\frac{4}{15}e^6+\frac{4}{45}e^8$

" (4). $+\frac{e^4}{3}-\frac{4}{15}e^6+\frac{4}{45}e^8$

" (5). $+\frac{e^4}{4}-\frac{1}{6}e^6+\frac{2}{45}e^8$

" (8). $+\frac{e^4}{8}-0-\frac{1}{30}e^8$

" (9). $+\frac{e^4}{32}+\frac{e^6}{8}-\frac{1}{12}e^8$

" (12). $+\frac{e^6}{8}-\frac{1}{24}e^8$

" (13). $+\frac{3}{64}e^6+\frac{1}{16}e^8$

Line (16). $+\frac{7}{64}e^5$

" (17). $+\frac{7}{128}e^5$

Sum $= +\frac{103}{96}e^4 - \frac{129}{320}e^6 + \frac{1673}{5760}e^8$

Mult. by $+1 \quad -\frac{1}{2}e^2 - \frac{1}{8}e^4$

$+\frac{103}{96}e^4 - \frac{129}{320}e^6 + \frac{1673}{5760}e^8$

$-\frac{103}{192}e^6 + \frac{129}{640}e^8$

$-\frac{103}{768}e^9$

$\therefore B_4 = +\frac{103}{96}e^4 - \frac{451}{480}e^6 + \frac{4123}{11520}e^8$

$i = -5.$

Line (2). $+\frac{125}{384}e^5 - \frac{3125}{9216}e^7 + \frac{78125}{516096}e^9$

" (3). $+\frac{125}{384}e^5 - \frac{3125}{9216}e^7 + \frac{78125}{516096}e^9$

" (6). $+\frac{25}{96}e^5 - \frac{375}{1536}e^7 + \frac{3125}{32256}e^9$

" (7). $+\frac{5}{32}e^5 - \frac{125}{1536}e^7 + 0$

" (10). $+\frac{1}{16}e^5 + \frac{25}{384}e^7 - \frac{125}{1536}e^9$

" (11). $+\frac{1}{80}e^5 + \frac{15}{128}e^7 - \frac{125}{1536}e^9$

" (14). $+\frac{5}{64}e^7 + 0$

" (15). $+\frac{7}{320}e^7 + \frac{5}{64}e^9$

" (18). $+\frac{5}{64}e^9$

" (19). $+\frac{9}{320}e^9$

Sum $= +\frac{1097}{960}e^5 - \frac{16621}{23040}e^7 + \frac{543513}{1290240}e^9$

Mult. by $+1 \quad -\frac{1}{2}e^2 - \frac{1}{8}e^4$

$\therefore B_5 = +\frac{1097}{960}e^5 - \frac{5957}{4608}e^7 + \frac{164921}{258048}e^9$

These examples are enough to show the course of the calculation. For every value of i the ratio of the coefficients in B_i in the first two lines is unity, and we have the same series in powers of e. As the denominators can be expressed in powers of 2, it may be convenient to introduce $\frac{e}{2}$, as *Hansen* has done. The following are the coefficients for $i = 8$, and $i = 9$.

$i = 8$.		$i = 9$.	
Line (1).	$+\frac{128}{315}e^8$	Line (2).	$+\frac{531441}{1146880}e^9$
" (4).	$\frac{128}{315}e^8$	" (3).	$\frac{531441}{1146880}e^9$
" (5).	$\frac{16}{45}e^8$	" (6).	$\frac{59049}{143360}e^8$
" (8).	$\frac{4}{15}e^8$	" (7).	$\frac{6561}{20480}e^9$
" (9).	$\frac{1}{6}e^8$	" (10).	$\frac{2187}{10240}e^9$
" (12).	$\frac{1}{12}e^8$	" (11).	$\frac{243}{2048}e^9$
" (13).	$\frac{1}{32}e^8$	" (14.)	$\frac{27}{512}e^9$
" (16).	$\frac{1}{128}e^8$	" (15).	$\frac{9}{512}e^9$
" (17).	$+\frac{1}{1024}e^8$	" (18).	$\frac{1}{256}e^9$
		" (19).	$+\frac{1}{2304}e^9$
Sum $= B_8 =$	$+\frac{556403}{322560}e^8$	Sum $= B_9 =$	$+\frac{10661993}{5160960}e^9$

Collecting results we have the series for the radius vector and the equation of center in terms of the mean anomaly and the excentricity to powers of e^9 inclusive.

$$\frac{r}{a} = 1 + \frac{1}{2}e^2 + \left[-e + \frac{3}{8}e^3 - \frac{5}{192}e^5 + \frac{7}{9216}e^7 - \frac{1}{81920}e^9\right]\cos nt$$
$$+ \left[-\frac{e^2}{2} + \frac{e^4}{3} - \frac{e^6}{16} + \frac{e^8}{180} - \ldots\ldots\ldots\right]\cos 2nt$$
$$+ \left[-\frac{3}{8}e^3 + \frac{45}{128}e^5 - \frac{567}{5120}e^7 + \frac{729}{40960}e^9\right]\cos 3nt$$
$$+ \left[-\frac{1}{3}e^4 + \frac{2}{3}e^6 - \frac{8}{45}e^8 + \ldots\ldots\ldots\right]\cos 4nt$$

$$+\left[-\frac{125}{384}e^5+\frac{4375}{9216}e^7-\frac{15625}{57344}e^9\ldots\ldots\right]\cos 5nt$$

$$+\left[-\frac{27}{80}e^6+\frac{81}{140}e^8\ldots\ldots\right]\cos 6nt$$

$$+\left[-\frac{16807}{46080}e^7+\frac{117649}{163840}e^9\ldots\ldots\right]\cos 7nt$$

$$+\left[-\frac{128}{315}e^8+\ldots\right]\cos 8nt+\left[-\frac{531441}{1146880}e^9\right]\cos 9nt$$

$$v-nt=\left[+2e-\frac{1}{4}e^3+\frac{5}{96}e^5+\frac{107}{4608}e^7+\frac{6217}{368640}e^9\right]\sin nt$$

$$+\left[+\frac{5}{4}e^2-\frac{11}{24}e^4+\frac{17}{192}e^6+\frac{43}{5760}e^8\ldots\ldots\right]\sin 2nt$$

$$+\left[+\frac{13}{12}e^3-\frac{43}{64}e^5+\frac{95}{512}e^7-\frac{973}{61440}e^9\ldots\right]\sin 3nt$$

$$+\left[+\frac{103}{96}e^4-\frac{451}{480}e^6+\frac{4123}{11520}e^8\ldots\ldots\right]\sin 4nt$$

$$+\left[+\frac{1097}{960}e^5-\frac{5957}{4608}e^7+\frac{164921}{258048}e^9\ldots\ldots\right]\sin 5nt$$

$$+\left[+\frac{1223}{960}e^6-\frac{7913}{4480}e^8\ldots\right]\sin 6nt+\left[+\frac{47273}{32256}e^7-\frac{1773271}{737280}e^9\right]\sin 7nt$$

$$+\left[+\frac{556403}{322560}e^8\ldots\ldots\right]\sin 8nt+\left[+\frac{10661993}{5160960}e^9\right]\sin 9nt$$

1905, March 22.

ON THE GENERAL APPLICABILITY OF "THE SHORT METHOD OF DETERMINING ORBITS FROM THREE OBSERVATIONS."*

A. O. LEUSCHNER.

The available solutions of the practical problem of determining from observations the heliocentric and geocentric paths of a comet or planet, when the attraction of the Sun alone is taken into account, leave, in the opinion of the great majority of astronomers, very little room for improvement. The principal methods are those of Newton, Olbers, Gauss, LaPlace, v. Oppolzer, Gibbs, Weiss and Harzer. Nearly all of them are products of the last century. Some of the methods are based on an assumption regarding the eccentricity, others are general.

* Read before the Section in Astronomy at the International Congress of Arts and Science, St. Louis, September, 1904, and at the February, 1905, meeting of the San Francisco Section of the American Mathematical Society.

In the case of a new comet or asteroid, it is customary to assume the eccentricity to be unity or zero, respectively, a more general solution being considered unwarranted on account of the shortness of the arc and the intricacy of the numerical operations involved. In general, these numerical operations increase with the accuracy aimed at in the first hypothesis. This is true also of Gibbs' Vector Method, which, in the first hypothesis, takes account of terms of the fourth order in the ratios of the triangles. No method is so arranged as to furnish directly from the residuals of the first hypothesis the corrections to the constants necessary to satisfy the observations. The applicability of many methods is limited by the existence of special or exceptional cases. The Short Method was intended to overcome as much as possible the difficulties referred to and to furnish the means of determining an orbit which would accurately satisfy the given observations, without previous hypothesis regarding the eccentricity and in less time than has been possible hitherto.

A general discussion of the convergency of the series involved in the Short Method would be of theoretical interest only, since in a given case all the terms which actually affect the solution are readily taken into account in the course of the calculation.

The fact, however, that applications to a two day and a twelve day arc in the case of Comet a 1904 (Brooks) yielded orbits which did not represent the later path of the comet so accurately as the preliminary parabolas derived in the usual way, seems to call for a discussion of the general applicability of the method.

On account of the great perspicuity of all the numerical operations involved in the Short Method it was readily recognized in the course of the calculation that in these cases definite elements could not be derived without hypothesis regarding the eccentricity. This was clearly expressed in publishing the results. For the two day arc as follows: "The elements indicate the general character of the orbit, but they are subject to considerable variation with a longer arc. The Short Method produces those elements which represent the observations exactly, and the results will vary more or less with slight changes in the observations."* And thus for the twelve day arc: "The new elements, also, are subject to some variation in the same direction as the first ones. They are true probably only to two decimals, the accuracy of the elements in the orbit being considerably less.

* L. O. Bulletin, No. 54.

The advisability of determining an orbit from a short arc without hypothesis as to the eccentricity might be questioned were it not for the fact that in applying this method the aim is to determine as rapidly as possible a finding ephemeris which accurately represents the given observations, the computation of the elements being incidental to conform to established custom. The fact that such an ephemeris, as far as comets are concerned, can thus be derived more conveniently than by means of a preliminary parabola is sufficient justification for departing from the old method. In many cases, the periodic character of an orbit is readily revealed from a short arc."*

The process of computing preliminary elements and from them an ephemeris for a comet or planet is in any case a form of extrapolation and subject to all its uncertainties. When a comet actually moves in a parabola, the uncertainties are at once removed by solving the orbit by one of the parabolic methods. But when it does not move in a parabola, the possibility of recognizing the periodic character of an orbit may be destroyed by solving for a parabola. I shall attempt to indicate how we may discriminate between the two cases in the Short Method and thus increase its general applicability.

In the Theoria Motus, Gauss calls attention to several cases to which his method is not applicable, viz.,

First, When one of the three geocentric places coincides with the corresponding heliocentric place of the earth, or is opposite to it;

Second, When the first geocentric place coincides with the third;

Third, When all three geocentric places and the second heliocentric place of the earth lie in the same great circle.

He continues to say that in the first case, the failure is due to the method, but in the second and third cases to the nature of the problem, the solution in any of the cases becoming very uncertain when any one of the conditions is approximately fulfilled. The first case occurs when at the time of one of the three observations, the body is near both a node and opposition or conjunction. Similar difficulties prevail in the general applicability of Oppolzer's method in which the first hypothesis takes account of terms of the third order in the ratios of the triangles, so that the geocentric distances are correct, inclusively of first order quantities.

The Short Method of Determining Orbits is based on LaPlace's and Harzer's methods in that it makes use of velocities and accelerations in the observed coördinates at the middle date. The

* L. O. Bulletin, No. 55.

immediate cause of the derivation of the Short Method was the failure of the older methods to produce an orbit from the limited number of observations available in the case of Asteroid 1900 GA. For this asteroid, the observations were not only situated in a great circle, but, furthermore, the object was discovered near opposition and near one of the nodes. The ease with which an orbit of the asteroid was derived by the Short Method promised a greater generality with smaller expenditure of time than the older methods. Theoretically, the method is applicable to comets as well as asteroids, and to furnish an illustration of the method in the case of a comet, an orbit was computed for Comet 1900 III by Messrs. R. H. Curtiss and C. G. Dall from two day intervals.* At the time when this computation was made, several parabolic orbits had appeared, while the result of the Short Method proved to be a periodic orbit with a mean daily motion of about 434″. Soon after, Professor Kreutz determined an orbit without hypothesis as to the eccentricity on the basis of an arc of about two weeks, and obtained a short period orbit agreeing very satisfactorily with the orbit determined by means of the Short Method.

In general, an accurate orbit cannot be determined from a very short arc without hypothesis as to the eccentricity, but with the ever increasing accuracy with which star places become known and observations are made, that part of the uncertainty in an orbit which is due to observation is constantly diminishing.† In view of this, the deviation of the orbits derived by the Short Method for Comet a 1904 from a parabola, which represents its motion very well, seemed to be larger than could be accounted for by the mere shortness of the arc and the inherent uncertainties of extrapolation, etc.

In this method, first approximations for the velocities and accelerations in α and δ at the time of the middle date are derived from the observations themselves, the effects of parallax and aberration being properly eliminated. The geocentric distance and the velocities in the heliocentric rectangular coördinates are computed next. These four quantities, together with the right ascension and declination at the middle date form the constants of the

* Publications of the Lick Observatory, Vol. VII, Part III.

† This is particularly true if, for a preliminary orbit, observations made by the same observer and the same telescope are used. In a discussion of the orbit of Encke's comet, Backlund has recently pointed out certain systematic differences between observations made with different instruments and at different intervals from the date of perihelion passage. (Memoires de L' Academie des Sciences de St Petersburg, VIII Serie, Volume XVI, No. 3.)

orbit. The first and third places are then reproduced by means of the chosen constants of the orbit, and the constants are corrected differentially by removing whatever residuals the first approximation may leave for the first and third places. In the computation of the orbit of Comet a 1904 derived from one day intervals, the first approximation left no residuals, the calculation being checked by reproducing the observations from the usual elements and constants to the equator, into which the original constants had been transformed. It will be shown, however, that in this case the observations can be represented by a variety of orbits.

It is a well known fact that, theoretically, in the case of the determination of an orbit from three observations, without hypothesis regarding the eccentricity, there can be but one or two solutions, no solution indicating an error in calculation or underlying positions; but that in certain cases, a group of practical solutions may lie near the theoretically possible solution hitherto has not been emphasized sufficiently, at least I am not aware of any attempts to determine the range of possible practical solutions about a theoretical one.

In order to determine the condition for the existence in a given case of a group of practical solutions, I have made use of my formulae 25 and 27 on page 12 of the "Publications of the Lick Observatory," Vol. VII, Part I.

They are as follows:

$$\partial\delta_{,} = B_{,}\,\partial\rho_0 - C_{,}\,\sin\delta_{,}\,\cos\alpha_{,}\,\partial x_0' - C_{,}\,\sin\delta_{,}\,\sin\alpha_{,}\,\partial y_0' + C_{,}\,\cos\delta_{,}\,\partial z_0'; \quad (1)$$

$$\partial\delta_{,,,} = B_{,,,}\partial\rho_0 - C_{,,,}\,\sin\delta_{,,,}\,\cos\alpha_{,,,}\partial x_0' - C_{,,,}\,\sin\delta_{,,,}\,\sin\alpha_{,,,}\partial y_0' + C_{,,,}\,\cos\delta_{,,,}\partial z_0'; \quad (2)$$

$$C_{,,,}\,\sin\alpha_{,,,}\partial\alpha_{,} - C_{,}\,\sin\alpha_{,}\partial\alpha_{,,,} = (A_{,}C_{,,,}\,\sin\alpha_{,,,} - A_{,,,}C_{,}\,\sin\alpha_{,})\partial\rho_0 + C_{,}C_{,,,}\,\sin(\alpha_{,,,} - \alpha_{,})\,\partial y_0'; \quad (3)$$

$$C_{,,,}\,\cos\alpha_{,,,}\partial\alpha_{,} - C_{,}\cos\alpha_{,}\partial\alpha_{,,,} = (A_{,}C_{,,,}\,\cos\alpha_{,,,} - A_{,,,}C_{,}\cos\alpha_{,})\partial\rho_0 + C_{,}C_{,,,}\,\sin(\alpha_{,,,} - \alpha_{,})\,\partial x_0'. \quad (4)$$

In these equations, the α and δ represent the positions computed from the first approximation, the $\partial\alpha$, $\partial\delta$ the residuals in the sense O — C, the A, B, and C auxiliary quantities, and the $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$ variations in the geocentric distance and in the velocities of the rectangular heliocentric coördinates at the middle date.

When the first and third places are accurately represented, the left hand members of these equations become zero. Then (3)

and (4) give $\partial y_0'$ and $\partial x_0'$ in terms of $\partial \rho_0$. On substituting $\partial y_0'$ and $\partial x_0'$ in terms of $\partial \rho_0$ in (1) and (2), we obtain two expressions for $\partial z_0'$ in terms of $\partial \rho_0$. If the ratios $\partial z_0'/\partial \rho_0$ resulting from these two expressions are different, then equations (1) to (4) can be satisfied only by $\partial \rho_0 = \partial x_0' = \partial y_0' = \partial z_0' = 0$, and the problem is determinate. If these ratios, however, are either identical or nearly so, then the problem is either indeterminate or admits only of the determination of the limits within which the elements may lie. In the latter case, the observations lead to a number of practical solutions actually satisfying the observations. The condition that such variations be possible may also be derived from formulae 26, loco. cit.

$$\begin{aligned} C_{,,,}\cos\delta_{,,,}\partial\delta_{,} - C_{,}\cos\delta_{,}\partial\delta_{,,,} = {} & + (B_{,}C_{,,,}\cos\delta_{,,,} - B_{,,,}C_{,}\cos\delta_{,})\partial\rho_0 \\ & - C_{,}C_{,,,}(\sin\delta_{,}\cos\delta_{,,,}\cos\alpha_{,} - \sin\delta_{,,,}\cos\delta_{,}\cos\alpha_{,,,})\partial x_0' \\ & - C_{,}C_{,,,}(\sin\delta_{,}\cos\delta_{,,,}\sin\alpha_{,} - \sin\delta_{,,,}\cos\delta_{,}\sin\alpha_{,,,})\partial y_0'; \quad (5) \end{aligned}$$

by placing the left hand member equal to zero and on the right hand side expressing $\partial x_0'$ and $\partial y_0'$ in terms of $\partial \rho_0$ as above. Then assuming for a moment that the observations are absolutely perfect, the condition that the orbit shall be indeterminate is that the coefficient of $\partial \rho_0$ be comparable in magnitude with the uncertainty involved in its numerical value.

If the coefficient exceeds in magnitude the uncertainty involved in its calculation, then a certain range of practical solutions will exist, whenever the number of places to which the coefficient is definitely known falls short of the number of places to which the orbit is being calculated. In practice, these conditions may be recognized at a glance. But since the observations are never perfect, it ought to be emphasized that the problem will also become indeterminate, if the coefficient of $\partial \rho_0$, although accurately known to several figures, becomes very small. In such cases, the smallness of the coefficient arises from the multiplication of several small but definitely known factors. Since the observations are affected by unavoidable errors, the absolute terms in equations (1) to (5) are in reality not zero, but differ from zero by a number of seconds which could be computed if the errors of observation were known. If c denote the coefficient of $\partial \rho_0$ then for an uncertainty of n″ in the absolute term of equation (5) the uncertainty in $\partial \rho_0$ and therefore in ρ_0 would be $\frac{n \sin 1''}{c}$. If this uncertainty becomes greater than about 0.3, a solution, without hypothesis regarding the eccentricity cannot be effected.

A discussion of the geometrical and physical meaning of indeter-

minate cases or of the existence of a group of practical solutions cannot be disposed of briefly, nor is such a discussion at all necessary for practical purposes, since in a given case the numbers themselves show clearly just what can be accomplished in solving the orbit from the given observations.

It may be of interest, however, to state without proof some of the conclusions that may be drawn in this connection:

First, The coefficient of $\partial\rho_0$ in (5) is independent of the solar coördinates, so that the constants may be corrected differentially even if the body is in conjunction or opposition and near its node. In this case, also, a first approximation to the constants is derivable in a simpler manner than in the ordinary case.

Second, Coincidence of the first and third places does not necessarily make the solution indeterminate.

Third, When the observations lie in a great circle, the solution becomes indeterminate only if three other conditions are satisfied, namely,

(a) When the intervals are equal,

(b) When the third powers of the intervals, multiplied by the solar constant k are numerically ineffective,

(c) When the heliocentric distances are comparatively large.

In this case, which can occur only with very short intervals, the coefficient of $\partial\rho_0$ in (5) becomes zero or very nearly so.

While thus in the older methods, the orbit becomes indeterminate, if computed without hypothesis as to the eccentricity, when the observations lie in a great circle, the orbit will become indeterminate in the Short Method only when, in addition, the foregoing other conditions are satisfied.

In this connection it may also be stated that the conclusion of Weiss that the observations can lie in a great circle, when r is not equal to R, only if the geocentric path has a point of inflexion at the middle date does not apply to very short intervals.

On examination, it was found that the first three observations of comet *a* 1904 presented a case of the third type cited above, the individual four conditions necessary to make the solution for $\partial\rho_0$ indeterminate being actually satisfied. This then, is a case in which the observations can be represented by an orbit of almost any period. The inaccuracy of the orbit deduced from the twelve day arc was due in the main to an error in one of the underlying positions.

In order to give the method a further test, Dr. Crawford of the University of California applied it to a forty day arc, using the observations of April 17, May 8, and May 24 on which the

parabola published in Lick Observatory Bulletin No. 56 by Dr. Aitken and Mr. Maddrill is based, and obtained a nearly parabolic orbit agreeing closely with the parabolic orbit deduced by the parabolic method. Numerical tests indicate that the small residuals of the parabola for the middle place account for the slight differences between the parabolic and nearly parabolic elements. Dr. Crawford's calculation was based on the second periodic orbit published in Lick Observatory Bulletin No. 55. A first approximation of the geocentric distance and heliocentric velocities for the middle place was determined from the preliminary ephemeris, the velocities being derived by numerical differentiation. On the basis of these four contants and the middle place, the residuals of the first and third places were computed. Then corrections to the geocentric distance and heliocentric velocities at the middle date were determined so as to remove the residuals for the first and third dates.

In this way, elements may readily be corrected on the basis of a longer arc without the necessity of deriving hypothesis after hypothesis, as would be the case in the older methods; without the necessity of performing a series of trials for the distances in each hypothesis; and without the necessity of performing for each triangular ratio a series of approximations in passing from one hypothesis to the next.

When the elements to be corrected have been computed by other methods and when the ephemeris data covering the new middle date are lacking, the starting values of the constants of this method may be computed without difficulty by formulae soon to be published.

The process just outlined for correcting elements readily admits of extension to any number of observations or normal places. While the Short Method has a more general applicability than the older methods, and while an orbit without hypothesis as to the eccentricity can be calculated in less time than it takes to compute a parabolic orbit by Olber's method, the fact that under the conditions referred to above an indeterminateness may exist, leading to a number of practical solutions within a certain range about the theoretical one, seems to call for modifications which will enable a computer to derive:

First, The range within which the period of the orbit may lie;

Second, A parabola directly, in case the parabola should be included in this range.

In practice, these modifications may be introduced in the course of the calculation, before the residuals of the first approximation

are removed or after an orbit has been obtained which actually satisfies the observations. Proceeding on the latter basis and making use of the well known relation:

$$g_0^2 = x_0'^2 + y_0'^2 + z_0'^2 = 2/r_0 - 1/a_0,$$

we may determine a system of variations in x_0', y_0', z_0', and r_0 which will change a_0 to a_1 without destroying the representation of the observations. Confining ourselves to linear variations, we have:

$$\frac{\partial r_0}{r_0^2} + x_0'\partial x_0' + y_0'\partial y_0' + z_0'\partial z_0' = \frac{1}{2a_0} - \frac{1}{2a_1}.$$

After introducing $\partial r_0 = \cos\beta\ \partial\rho_0$ and replacing also $\partial x_0'$, $\partial y_0'$, and $\partial z_0'$ in terms of $\partial\rho_0$ we obtain an expression for the value of $\partial\rho_0$ which corresponds to a change of a_0 into a_1.

The maximum positive and negative values of $\partial\rho_0$ feasible without destroying the representation of the observations are to be obtained from the coefficient of $\partial\rho_0$ in (5). The limiting periods of a group of practical solutions are determined by substituting the maximum and minimum feasible variations $\partial\rho_0$ in the foregoing equation and solving for the corresponding two values of a_1.

If the value of a_1 corresponding to the maximum or minimum $\partial\rho_0$ becomes infinite or very large, then the observations can be represented by a parabola or by a very nearly parabolic orbit. In that case, we let $\frac{1}{2a_2} = 0$ and solve for $\partial\rho_0$, etc. This course would be taken whenever it is desired to compute a parabola, whether it be included in the group or not.

From $\partial\rho_0$ the $\partial x_0'$, $\partial y_0'$, $\partial z_0'$, and ∂r_0 are then readily calculated. If we let $a_1 = \infty$, a system of variations may thus be determined which will make the orbit a parabola. In the form in which the problem has been treated, there will be two results for $\partial z_0'$, in case the parabola is not definitely included in the group of solutions. In such cases, the residuals due to the deviation of the orbit from a parabola may be distributed between the first and last declinations. It is readily seen, however, that the formulae may be arranged so as to express the deviation in residuals of any one or more places. In practice, it will be well to always determine a parabola in cases where the parabola is approximately included in the group of possible solutions. As an illustration of these principles, Dr. Crawford has determined the variations necessary to pass a parabola through the forty day arc in the case referred to above, of Comet a 1904, and his results are practically identical with those derived by the ordinary para-

bolic method. The residuals were distributed between the first and last declinations, and are as follows:

$$0 - C = -2.''5; \quad = -1.''1.$$

The principle has also been applied with success to the first three observations for the purpose of changing the original short period orbit from one day intervals to a parabola.

It may be expected that with the modifications referred to here in a general way and soon to be published more fully, the Short Method will yield satisfactory results in every case.

Note: Since writing the foregoing paper, the Author has derived a further modification of the Short Method in order to adapt the same to the direct computation of a parabola. The new method was applied with success to the observations of March 26, 27, 30, of Comet a 1905 (Giacobini) by Dr. Russell Tracy Crawford and Mr. James D. Maddrill.

UNIVERSITY OF CALIFORNIA,
BERKELEY ASTRONOMICAL DEPARTMENT.

MARS.

WILLIAM H. PICKERING.

FOR POPULAR ASTRONOMY.

In the April number of POPULAR ASTRONOMY attention was called to this planet, and the opinion expressed that the polar caps should appear during the month. Owing to the press of other matters it was not possible to maintain visual observations in Cambridge, but photographs were obtained under the direction of Mr. King with the 11-inch Draper telescope whenever practicable. An enlarging lens was employed, giving the original negatives on a scale of about 2.″5 to the millimeter.

The first photograph was taken upon March 31, others being secured upon April 1, 2, 8, 15, 16, 18, 23, 25, 27, and 30. The first photograph showed clouds at both the limb and terminator, but no polar caps properly so-called appeared until April 23, when a large light area was clearly visible at the south pole. It did not appear dark enough for snow however, but more resembled an extensive cloudy region. It has remained visible upon the photographs since that date, although with slightly diminished intensity and size. A minute light area appeared near the north pole on April 15, but was seen only with difficulty.

A visual examination of Mars was made on the night of April 30, with the 24-inch reflector. The southern polar cap was clearly visible, extending far to the north in longitude 340°, but

its intensity was slight, little exceeding that of the limb in other regions. It is probable that when the Martian clouds clear away, snow will be found lying in their place.

The heliocentric colongitudes of the planet upon April 15 and 23 were 216° and 220°. These positions would correspond on the Earth to August 3, and 7, or to near the end of the winter of the southern hemisphere. Snow seldom comes earlier upon Mars. Extensive green areas should be visible in June, and the Mare Erythraeum recently described by Professor Lowell as brown, should by that time have recovered its normal color. This change of color with the seasons seems to the writer the best proof of the existence of vegetation upon Mars.

HARVARD OBSERVATORY, May 2, 1905.

PIETRO TACCHINI.

HECTOR MACPHERSON, JUNIOR.

FOR POPULAR ASTRONOMY.

The death of Professor Tacchini, which took place at Modena on March 24th last, removes from the scene one of the foremost astronomers of the present day. Not only Italy, but the entire scientific world, has sustained a severe loss.

Born at Modena on March 21st, 1838, Pietro Tacchini received his education in the famous University of his native town, where he took his degree in 1857, at the age of nineteen. In the following year he studied astronomy under the illustrious Giovani Santini in the Observatory of Padua and in 1859, on the formation of the Kingdom of Italy, was appointed director of the Observatory at Modena. Through the influence of Father Secchi he was appointed in 1863 astronomer in the Observatory at Palermo, at that time under the direction of Professor Cacciatore, and at Palermo he commenced his observations on the Sun, both direct and by means of the spectroscope.

In 1868 M. Janssen and Sir Norman Lockyer discovered that the spectra of "red flames" of the Sun or solar prominences could be observed in daylight and the following year Sir William Huggins found that by widening the slit of the spectroscope the forms of the prominences could be observed. The famous astronomers of Italy, Secchi and Respighi, at once took up the study and in 1870 Signor Tacchini commenced daily observations of the prominences at Palermo. For more than thirty years he continued his observations. The prominences were his chief ob-

jects of study. He did for the observation of the prominences what Schwabe did for the spots, and many remarkable discoveries resulted from his skill and perseverance. In 1870 he observed the total eclipse in Sicily and in 1874 he visited India to observe the transit of Venus. After the death of Secchi in 1878 and the retirement of his successor Ferrari in 1879, Signor Tacchini was promoted to the post of director of the Observatory of the Collegio Romano and in the same year he organized at the request of the Italian Government, the Meteorological and Geodynamical Institute of Rome. In 1883 he accompanied M. Janssen to Caroline Island, a coral reef in the Pacific, for the purpose of observing the total solar eclipse on May 6th of that year. "Seldom," says Miss Clerke, "has a more striking proof been given of the vividness of human curiosity as to the condition of worlds outside our own, than in the assemblage of a group of distinguished men from the chief centres of civilization, on a barren ridge, isolated in a vast and tempestuous ocean, at a distance in many cases of 11,000 miles and upwards from the ordinary scene of their labors."

In 1872 Signor Tacchini founded the now famous Society of Italian spectroscopists, an organization which has placed Italians in the front rank of solar observers, as the magnificent work of Tacchini, Secchi, Respighi, Ricco, Lorenzoni and Mascari testifies. But the most important work was accomplished by Tacchini himself. As early as 1873 his studies of the solar prominences led to important results as to the connection between the aurora borealis and the Sun. He pointed out that "the terrestrial auroras are more in relation with the protuberances than with the spots;" and from observations of the prominences he was enabled to predict the appearance of several auroras which would not otherwise have been observed at Palermo. Some of his individual observations were of surpassing interest. For instance on January 30, 1885, he measured at Rome one of the greatest prominences which has ever been observed, no less than 142,000 miles in height—eighteen times the diameter of the Earth. He was the first to notice that the coronal streamers originate in regions where the prominences are most numerous. At the sun-spot maximum, prominences are spread all over the Sun and the corona distributed uniformly; while at the minimum the prominences are withdrawn from the polar regions, which at the same time become devoid of coronal streamers. He also pointed out that the district of what Miss Clerke calls "greatest prominence-frequency" overlaps by several degrees that of spot-frequency.

The eleven-year period of solar spots governs the distribution of prominences, but the prominence-maximum lasts longer than the spot-maximum.

Much of Professor Tacchini's solar work was accomplished during total eclipses. On the occasion of the eclipse of 1883 he discovered white prominences, giving a continuous spectrum. This discovery was confirmed by subsequent observations at the eclipse of 1886, when Professor Tacchini observed a gigantic white prominence, 150,000 miles in height giving a continuous spectrum with bright calcium lines; and by a comparison of the forms and sizes of prominences during the eclipse and after the end of totality he demonstrated that ordinarily the spectroscope reveals only the cores, glowing and vaporous of these objects. The existence of purely white prominences was explained by the famous Italian astronomer by the fact that these objects do not possess any gaseous cores, which distinguish the ordinary solar prominences. In a letter to the writer of this article, received last year, he expressed the view that many of the phenomena of the chromosphere and solar atmosphere are the products of electrical action on the Sun.

During 1895 Professor Tacchini attentively studied Venus to determine the rotation period; and his observations were distinctly confirmatory of those of Professor Schiaparelli which indicated a period of 225 days, equal to the period of revolution. Professor Tacchini was the founder of the Seismological Society of Italy—devoted to the study of earthquakes—and the observatories of Cantania and Mount Etna. He was a Cavalier of the Civil Order of Savoy, Member of the Astronomische Gesellschaft and the Société Astronomique de France. In 1883 he was elected an Associate of the Royal Astromical Society and in 1888 received the Rumford Medal of the Royal Society and in 1892 the Janssen Medal of the French Academy of Sciences. In 1901 he resigned the directorship of the Observatory of the Collegio Romano and retired to his house at Modena, Cá di Soli, where he resided until his death.

Although Pietro Tacchini's life-work is completed, his reputation is a never-fading one. He combined extraordinary perseverance and patience with brilliancy and originality, and this was the secret of his success as a solar observer. He ranks as one of the greatest observers of the Sun and one of the leading astronomers not only of Italy, but of the world. His name will live forever in the annals of the science which he made peculiarly his own.

AN INTRODUCTION TO THE DIFFERENTIAL CALCULUS, BY MEANS OF FINITE DIFFERENCES. II.

COPYRIGHT BY ROBERDEAU BUCHANAN, 1905.

FOR POPULAR ASTRONOMY.

6. *Graphical Representation.* In Fig. 1 draw the axis of X a horizontal line, and the axis of U perpendicular to it at the point O. Lay off the negative values of x on the left and the positive on the right. The spaces between these points are each dx which is taken as unity in the example column 1. Upon these points erect the ordinates and lay off the values of u resulting from the values of x in example 1. These latter points a b c etc. may be connected by a curve which represents the equation 10 or 11.

7. If from the points a b c, etc., already found we draw the horizontal lines aa', bb', cc', etc., to the next ordinate on the right side (which is the positive direction of the motion of x, from left to right,) the lines o'a a'b b'c, etc., are the finite differences, belonging to the values of u. These lines form a familiar figure in most works on the calculus, but which is usually not sufficiently explained, beyond a mere statement. The angles which the chords Oa ab bc, etc., make with the axis of X are expressed by the tangents

$$\frac{ao'}{o'O} \quad , \quad \frac{ba'}{a'a} \qquad \frac{cb'}{b'b} \quad , \text{ etc.} = \frac{\triangle u}{\triangle x}$$

which are very nearly the tangents to the middle points of the curve; but not exactly because the curvature is different at the two ends of the chords. The differential coefficient for an ordinate midway between those in the figure takes account of the curvature, and gives the tangents to these middle points. It will be noticed that on the left of the axis of U, du is laid off downwards, or in a negative direction, so that the tangent is a negative quantity, showing that the curve is moving in the negative direction, or downwards, in the figure. And $\frac{d^2u}{dx^2}$ is positive; which also shows that the curve is convex towards the axis of X, Fig. 1.

8. If in Equation (10) we give the negative sign to the right member

$$u = -x^2 \tag{13}$$

we have

$$\left.\begin{aligned} \frac{du}{dx} &= -2x \\ \frac{d^2u}{dx^2} &= -2 \end{aligned}\right\} \tag{14}$$

Which is shown as follows,

EXAMPLE 1, biz:

x	u	$\triangle^1$	$\triangle^2$
-2	-4		
		$+3$	
-1	-1		-2
		$+1$	
0	0		2
		-1	
$+1$	-1		2
		3	
2	4		2
		5	
3	9		2
		7	
4	16		-2
		-9	
$+5$	-25		

And comparing with Example 1, it is seen that the effect is to change the signs of all the difference. The second differences being negative, show that the curve has a maximum point and that it is concave to the axis of x (that is concave to a line below the curve); and graphically, Fig. 1, it has been revolved round the axis of X 180° and lies wholly below it. The effect upon the Finite Differences is also seen. For $x = +3$ u becomes -9, and the first difference (below the line as usual,) is -7.

9. *A constant connected with a variable by the sign plus or minus disappears in differentiation*, but the curve is moved along the axis of U. In the general equation (9), let $a = 2$ a constant, thus we have

$$\left.\begin{aligned} u &= 2 + x^2 \\ \frac{du}{dx} &= 2x \end{aligned}\right\} \qquad (15)$$

With this equation form the finite differences as in Example 2. It is seen that although the column u is not the same as in Ex-

EXAMPLE 2, $u = 2 + x^2$ (15)

x	u	$\triangle^1$	$\triangle^2$
-5	$+27$		
		-9	
4	18		$+2$
		7	
3	11		2
		5	
2	6		2
		3	
-1	3		2
		-1	
0	2		2
		$+1$	
$+1$	3		2
		3	
2	6		$+2$
		5	
$+3$	$+11$		

ample 1, yet the differences are identical, since the constant cancels out in the subtraction. In other words we have a portion of the example as follows:

$$\begin{array}{ll} x=-5, & u=+25+2 \\ x=-4, & u=+16+2 \\ x=-3, & u=+\ 9+2 \end{array}$$

when u is seen to be composed of two terms the first being identical with Example 1, and the other a constant. The first term gives of course the same differences as in Example 1 but the second term has no differences. That is a constant term has no differential and disappears in differentiation as it does in Finite Differences. If we plot the numbers in Example 2 on Fig. 1, we find that it is similar to that figure, but raised two units along the axis of U or what is the same thing it will be the curve of Example 1 if the axis of X should be drawn through the point P. The effect of this independent constant is thus not to alter the curve itself, but to place it differently upon the axes.

10. We see also that when integrating we must always affix the letter C to the result to allow for a constant that may have disappeared in differentiating.

11. *Effect of a constant connected with the function of the variable.* The curve moved along the axis of X. In the general equation 9, let $b = +2$

Then

$$\left.\begin{array}{l} u = (x+2)^2 = x^2+4x+4 \\ \dfrac{du}{dx} = 2x+4 \end{array}\right\} \qquad (16)$$

This equation with its differences is shown in Example 3, where u is seen to differ from Example 1 by a variable quantity, but the

EXAMPLE 3, $u = (2+x)^2$ (16)

x	u	$\triangle_1$	$\triangle_2$
−5	+9		
		−5	
4	4		+2
		3	
3	+1		2
		−1	
2	0		2
		+1	
−1	+1		2
		3	
0	4		2
		5	
+1	9		+2
		+7	
+2	+16		

differences are still the same indicating the same curve as before;

but still differently placed upon the axes, that is, moved along the axis of X, two units.

12. *Explanation of two forms of an equation resulting from Integration, differing by a constant.* If we Integrate equation (16) viz.

$$\frac{du}{dx} = 2x + 4$$

we have

$$u = \int (2x + 4)\, dx = x^2 + 4x + C \qquad (17)$$

We cannot determine the value of the constant C from this equation alone, for if $x = 0$, $C = 0$. It is seen that the constant 4 moves the curve 4 units along the axis of U and the constant 2 also moves it 2 units along the axis of X. Fig. 1. The student may form the differences and test this. We can, however, integrate this equation in another way by factoring out the constant 2, thus

$$u = \int (2x + 4) = \int (x + 2)2dx = (x + 2)^2 + C \qquad (18)$$

If $x = 0$, $C = -4$ which reduces this equation to the previous form; but if C is taken as zero (for the constants are generally arbitrary) we have the original equation (16). It is thus seen that integration may teach us something, by giving us a form of equation we did not previously know. And moreover the works on the Calculus do not generally explain this point sufficiently,—why a figure or quantity may be summarily dropped or how a constant mysteriously appears; But when we know that these quantities refer only to the position of the curve upon the coördinate axes, and that the position of the axes is arbitrary, then the matter becomes clear at once.

13. *A constant factor of the variable appears in the differences.* All the differences are affected proportionally. The curve is spread out or contracted.

In the general equation 9 let $c = \frac{1}{2}$. Thus

$$\left.\begin{aligned} u &= \frac{1}{2}x^2 \\ \frac{du}{dx} &= \frac{1}{2}2x = x \\ \frac{d^2u}{dx^2} &= 1 \end{aligned}\right\} \qquad (19)$$

EXAMPLE 4. $u = \frac{1}{2}x^2$. (19.)

x	u	$\triangle_1$	$\triangle_2$
−4	+8.0		
		−3.5	
3	4.5		+1.0
		2.5	
2	2.0		1.0
		1.5	
−1	0.5		1.0
		−0.5	
0	0.0		1.0
		+0.5	
+1	0.5		1.0
		1.5	
2	2.0		1.0
		2.5	
3	4.5		+1.0
		+3.5	
+4	+8.0		

The differences of this equation are given in Example 4. The effect of this constant factor is seen to be, that all the differences are one half of the corresponding values in Example 1; that is in the words of the calculus "a constant factor appears in the differentials." (Courteney) The curve of this last equation is shown in Fig. 1 by a series of dots upon the ordinates. The effect is to spread out the curve, or flatten it toward the axis of X. If the constant factor had not been a fraction, but an integer 2, 3, etc., the curve would have been compressed,—the branches drawn together.

14. *A change in the value of the independent variable, alters the interval of the computed values.* We will resume equations 10 and 11 of the first example namely:

$$\left.\begin{aligned} u &= x^2 \\ \frac{dx}{dx} &= 2x \end{aligned}\right\} \qquad (20)$$

We also have $\left.\dfrac{d^2u}{dx^2} = 2\right\}$

As the value of dx is arbitrary, it is usually as a matter of simplicity taken as unity; or in other words whatever the actual interval may be it is considered as a unit. This unit may be one day or one hour or a period of 20 days, etc. When interpolating a function the interval of computed dates is always considered as a unit. In this example, however, we will take this interval as one half, that is, $dx = \dfrac{1}{2}$.

Then the above equations become

$$\left.\begin{aligned} du = 2x\,dx \quad &= 2x \cdot \frac{1}{2} \quad = x \\ d^2u = 2\,dx^2 \quad &= 2 \cdot \left(\frac{1}{2}\right)^2 = \tfrac{1}{2} \end{aligned}\right\} \qquad (21)$$

These equations are shown in Example 5.

EXAMPLE 5. EQUATION (21).

$$u = x^2 \qquad du = \frac{1}{2}\,.$$

x	u	Δ^1	Δ^2
−2	+4.00		
		−1.75	
1.5	2.25		+0.50
		1.25	
1.0	1.00		+0.50
		0.75	
−0.5	0.25		0.50
		−0.25	
0.0	0.00		0.50
		+0 25	
+0.5	0.25		0.50
		0.75	
1.0	1.00		0 50
		1.25	
1.5	2 25		+0.50
		1.75	
+2.0	+4.00		

If we should take the interval as one-fourth, then we should have

$$\left.\begin{aligned} \frac{du}{dx} &= 2x\,dx = 2x\,.\,\frac{1}{4} = \frac{1}{2}\,x \\ \frac{d^2u}{dx^2} &= 2dx^2 = 2\,.\,\frac{1}{16} = \frac{1}{8} \end{aligned}\right\} \qquad (22)$$

15. It is seen from these equations that by diminishing the interval of the computed dates that the first difference is diminished $\frac{1}{n}$ the second differences $\left(\frac{1}{n}\right)^2$ the third differences $\left(\frac{1}{n}\right)^3$ and so on.

In any computation therefore if we find there are higher differences that are inconveniently large for interpolation or other purposes we can cause them to become less or practically to vanish by diminishing the interval of the computed quantities.

16. *Particular values of an equation.* Maxima and Minima. The rule in the calculus for this is to differentiate and place the first differential coefficient equal to zero, and solve the equation with respect to x. If x results a finite value or zero it indicates one of these two. This value of x substituted in the original equation, gives its maximum or minimum value. To ascertain which one it is, differentiate a second time, and if the value of x found above makes the second differential coefficient positive, the point is a minimum, but if the sign results negative, the point is a maximum.

To illustrate this we will take equation (11) of Example 1, (for we have not yet exhausted this little equation.)

$$\left.\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x = 0 \quad \text{Then } x = 0 \\ \frac{d^2u}{dx^2} &= +2 \qquad \text{positive value} \end{aligned}\right\} \qquad (23)$$

The first differential coefficient when placed equal to zero and solved gives $x = 0$, and this substituted in the first of these equations gives $u = 0$.

The second differential coefficient has the positive sign which shows that the value $u = 0$ is a minimum.

This same process can be made use of with the differences in Example 1. We here look for a point of the curve which is parallel to the axis of X because $\frac{du}{dx}$ being the tangent of the angle which the curve makes with the axis of X, if there is a minimum point the curve after descending turns and then ascends, and the lowest point is when the curve is parallel to the axis of X. Its tangent line being parallel, the angle must be zero and the trigonometrical tangent is zero or the first difference $\Delta_1 = 0$.

We therefore examine the column of 1st Differences for the value 0 opposite which we have the values of u and x. In some equations, the curve may approach the axis of X and become parallel to it but yet contain no maximum or minimum, $\frac{d^2u}{dx^2}$ being 0 is neither + nor —. This is the case with the cubical parabola given by the equation $u = x^3$. The origin is a point of inflexion.

17. *Points of Inflexion.* It was remarked in Article 7 that the curve Fig. 1 was convex to the axis of X in which case $\frac{d^2u}{dx^2}$ is positive. When the curve is concave to the axis $\frac{d^2u}{dx^2}$ is negative. At the point of inflexion therefore $\frac{d^2u}{dx^2}$ must change its sign, which it cannot do unless it becomes 0 or ∞. Hence these values characterize a point of inflexion.

$$\frac{d^2u}{dx^2} = 0 \qquad \text{or} \qquad \frac{d^2u}{dx^2} = \infty \tag{24}$$

The example in Article 19 $u = x^5$ shows a point of inflexion when $x = 0$. A simpler example is given in the cubical parabola $u = x^3$ which may be computed, differenced, and plotted, by the student.

18. *Cusps.* These are indicated according to the Calculus by $\frac{du}{dx}$ having two equal and real values. Thus in the semi-cubical parabola,

$$\left.\begin{aligned} u^2 &= x^3 \\ u &= \sqrt{x^3} \end{aligned}\right\} \tag{25}$$

Solving this equation for various values of x we have the following values and differences:

x	u	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\triangle_4$
−2					
		0.000			
−1			0.000		
		0.000		∓1.000	
0	0.000		∓1.000		±1.172
		∓1.000		±0.172	
1	∓1.000		0.828		±0.116
		1.828		0.288	
2	2.828		0.540		∓0.184
		2.368		0.104	
3	5.196		0.436		∓0.044
		2.804		±0.060	
4	8.000		∓0.376		
		∓3.180			
5	∓11.180				

There are no real values for the negative values of x, and for $x = 0$ the first real value $x = 0$ appears; for each positive value of x we have two real and equal values with contrary signs,—in fact two curves. Hence $u = 0$ is a cusp-point. The form of the curve, the reader can doubtless find in any work on the Calculus. Δ_2 having the double sign the value $u = 0$ is a maximum for one branch and a minimum for the other.

19. *The Distinction between a Differential and a Difference rigorously shown.* Professor Chauvenet in his Spherical and Practical Astronomy has deduced a series of formulae for transforming a series of Finite Differences into Differential coefficients. These formulae are similar to the usual formulae for interpolation, both being derived from the algebraic formula for the sum of the series. In the present formulae in order to interpolate for values lying in a horizontal line with the primitive, the even differences, and the mean of the odd differences are employed; but in the general formula for interpolation, where the value lies between the given primitive and the next following, the odd differences and the mean of the even are employed.

These formulae which may also be found in Watson's *Theoretical Astronomy* are as follows,—changing Chauvenet's notation slightly to conform to that of the present work.*

$$\left.\begin{aligned}
\frac{du}{dx} &= \frac{1}{w}\left(\frac{\Sigma\triangle_1}{2} - \frac{1}{6}\frac{\Sigma\triangle_3}{2} + \frac{1}{30}\frac{\Sigma\triangle_5}{2} - \text{etc.}\right)\\
\frac{d^2u}{dx^2} &= \frac{1}{w^2}\left(\triangle_2 - \frac{1}{12}\triangle_4 + \text{etc.}\right)\\
\frac{d^3u}{dx^3} &= \frac{1}{w^3}\left(\frac{\Sigma\triangle_3}{2} - \frac{1}{4}\frac{\Sigma\triangle_5}{2} + \text{etc.}\right)\\
\frac{d^4u}{dx^4} &= \frac{1}{w^4}\left(\triangle_4 - \text{etc.}\right)\\
\frac{d^5u}{dx^5} &= \frac{1}{w^5}\left(\frac{\Sigma\triangle_5}{2} - \text{etc.}\right)
\end{aligned}\right\} \qquad (26)$$

* The author has employed differences extensively in computations of the planets for the Nautical Almanac, U. S. Naval Observatory. The hourly motions printed in the Nautical Almanac, and the aberration are computed by means of the first equation of formula (26). The formulae are also used for correcting the elements of orbit of a planet from three or more observations. In this way, more than twenty-five years ago, he discovered this method of explaining the Calculus.

In these equations the notation $\frac{\Sigma \Delta}{2}$ denotes the mean of the adjacent odd differences, lying above and below the line D in Example 6. The factor w is used if we wish to reduce the differential to a lesser interval than that of the tabulated values. As we here use the formulae it is unity, and neglected.

20. In this article we will assume the equation

$$u = x^5 \tag{27}$$

with which we form the following tabulated values with their differences, Example 6 as follows:

EXAMPLE 6. $u = x^5$, EQUATION 27.

x	u	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\triangle_4$	$\triangle_5$	
−3	− 243						
		+ 211					
2	32		− 180				
		31		+ 150			
−1	− 1		− 30		−120		
		+ 1		30		+120	
0	0		0		0		
		+ 1		30		120	
+1	+ 1		+ 30		+120		
		31		150		120	
2	32		180		240		
		211		390		120	
3	**243**		**570**		**360**		**D**
		781		750		120	
4	1024		**1320**		480		
		2101		**1230**		120	
5	3125		2550		**+600**		
		4651		+1830		**+120**	
6	7776		+4380				**F**
		+9031					
7	+16807						

For the numerical values to be used we assume

$$x = +3 \tag{28}$$

and for this value the differential coefficients lie in the horizontal line from 3^5 to D, while the Finite Differences lie in the diagonal from 3^5 to F.

The differential coefficients will now be formed, and their values for $x = +3$ computed, as follows:

$$\left.\begin{aligned} \frac{du}{dx} &= 5x^4, & \text{and when } x = 3 \quad &= 405 \\ \frac{d^2u}{dx^2} &= 20x^3, & &= 540 \\ \frac{d^3u}{dx^3} &= 60x^2 & &= 540 \\ \frac{d^4u}{dx^4} &= 120x, & &= 360 \\ \frac{d^5u}{dx^5} &= 120, & &= 120 \end{aligned}\right\} \tag{29}$$

Now substituting in equation (26) the mean of the odd differences, and the values of the even differences lying along the horizontal line D in Example 6, we have

$$\left.\begin{aligned}
\frac{du}{dx} &= \frac{992}{2} - \frac{1}{6}\,\frac{1140}{2} + \frac{1}{30}\cdot\frac{240}{2} = 496 - 95 + 4 = 405\\
\frac{d^2u}{dx^2} &= 570 - \frac{1}{12}\,360 \qquad\quad = 570 - 30 \quad = 540\\
\frac{d^3u}{dx^3} &= \frac{1140}{2} - \frac{1}{4}\,\frac{240}{2} \qquad = 570 - 30 \quad = 540\\
\frac{d^4u}{dx^4} &= 360 \qquad\qquad\qquad\qquad\qquad\quad = 360\\
\frac{d^5u}{dx^5} &= 120 \qquad\qquad\qquad\qquad\qquad\quad = 120
\end{aligned}\right\} \qquad (30)$$

which values agree with the differential coefficients found above in equation 29, by differentiation.

21. It is further and clearly seen that one distinction between a differential coefficient and a Finite Difference is one of Position in the scheme of tabulated Finite Differences. And a Differential Coefficient of any order may be correctly defined as that certain interpolated value of a series of tabulated Finite Differences of the same order, which lies in a horizontal line with its primitive function; while the Finite Differences are those values obtained by subtraction which lie in the downward diagonal from the same primitive function.

22. *Further criticisms on the Method by Limits.* We see moreover from the first equation of 29 that when applied to a function of the second power there is but one term.

$$\frac{du}{xd} = \frac{1}{2}\,\Sigma\,\Delta_1 = \begin{cases}\Delta_1\text{ (upper value)} + \tfrac{1}{2}\,\Delta_2\\ \Delta_1\text{ (lower value)} - \tfrac{1}{2}\,\Delta_2\end{cases} \qquad (31)$$

which by the differences of example 1 when $x = 3$ becomes

$$\frac{du}{dx} = \Delta_1 - \frac{1}{2}\,\Delta_2 = 7 - 1 = 6$$

Again

$$u = x^2 \qquad \frac{du}{dx} = 2x,\ (\text{when } x = 3) = 6$$

We now discover what becomes of the term h in equation (4)

$$u_1 - u = 2xh + h^2$$

in which h must be taken equal to unity to correspond with the interval in Example 1. It does not "disappear" by becoming less and less, but is controlled or cancelled by the term $\frac{1}{2}\,\Delta_2$ in equation (26) which is 1 in equations of the second degree, (provided the coefficient of x in the primitive equation is unity.)

23. We are now prepared to say that in such equations as (7)

which we have been considering, that for any given value of x both $u_1 - u$ and $2x$ are constants, (with coefficients however,) which becomes apparent by examining the changes which take place while h is supposed to decrease without limit. We will examine both equations (4) and (7) as follows:

	Equation (4).	Equation (7).
	$u_1 - u = 2xh + h^2$	$\frac{u_1 - u}{h} = 2x + h$
When $x = 1$,	$u_1 - u = 2x + 1$	$(u_1 - u) = 2x + 1$
$= \frac{1}{2}$,	$u_1 - u = \frac{2x}{2} + \frac{1}{4}$	$2(u_1 - u) = 2x + \frac{1}{2}$
$= \frac{1}{4}$,	$u_1 - u = \frac{2x}{4} + \frac{1}{16}$	$4(u_1 - u) = 2x + \frac{1}{4}$
$= \frac{1}{8}$,	$u_1 - u = \frac{2x}{8} + \frac{1}{64}$	$8(u_1 - u) = 2x + \frac{1}{8}$
Limit, 0,	$u_1 - u = 0 + 0$	$\infty(u_1 - u) = 2x + 0.$

These equations are convertible to one another as they should be; in the first the quantity $2x$ does not remain to reach the limit as fondly hoped in this method; but becomes equal to zero; the term $(u_1 - u)$ also becomes equal to zero and there is nothing left but the primitive equation. This is the correct and legitimate effect of diminishing the value of h. In the other equation although the quantity $2x$ remains yet the coefficient of the left member becomes infinite, and at the limit $(u_1 - u)$ takes the indeterminate form $\frac{2x}{\infty}$; an inconsistency arising from neglecting the fundamental principle of Finite Differences, that h must be a constant.

24. In Haddon's "Examples in the Differential Calculus," Weales Series London, the above mentioned inconsistencies are very clearly shown. Assuming the equation $u = x^3$ the author proceeds, "$\frac{u_1 - u}{h} = 3x^2 + 3xh + h^2 =$ Ratio of increment of function to increment of variable. Now the first term of this expression for the ratio being $3x^2$, it is obvious that h may undergo any change of value whatever, without affecting this first term. Let h then continually decrease in value until it is $= 0$."

May it? Most assuredly it may; but this sophistry keeps out of sight that the first member becomes infinite from which nothing can then be proved.

We see that in the first member of all these similar equations,

that $(u_1 - u)$ does not suffer much decrease in value. In fact it cannot decrease smaller than the first term of the second member, while its denominator h decreases without limit.

There seems to be also another peculiarity in these equations. They start with x as a variable receiving the increment h, but when $(u_1 - u)$ is reached x seems to be regarded as a constant with h a variable!

The true nature of the change which this method by limits endeavors to explain is shown in Equations (26) to (30) where it is seen that all the terms of the higher differences combine in certain proportions by interpolation and are numerically added to or subtracted from the first term (which remains unchanged) and thus make the change in the first member of the equation, while this new numerical value is still denoted by the quantity $3x^2$ in the present equations and $2x$ in those on the previous pages. This is an intricate point and sophistry can make much out of it. In one sense $3x^2$ is constant and in another sense it has changed its numerical value. The numerical value depends upon whether we regard $3x^2$ as a term of a Finite Difference or a Differential Coefficient. How easy to slide from one to the other! not knowing or suppressing the fact that the numerical values are different.

Finally. If we have a quantity x and add to it another quantity h and let h diminish until it becomes zero, what in the name of common sense have we left but the original quantity?

25. *The reverse problem. Finite Differences derived from Differential Coefficients.* Professor George Boole has given in his Calculus of Finite Differences* the following formula which is the fundamental relation between finite differences and differential coefficients:

$$\left.\begin{aligned}\triangle^n u = \frac{d^n u}{dx^n} + \frac{\triangle^n 0^{n+1}}{1.2.(n+1)} \cdot \frac{\triangle^{n+1} u}{dx^{n+1}} \\ + \frac{\triangle^n 0^{n+2}}{1.2.(n+2)} \cdot \frac{d^{n+2} u}{dx^{n+2}} + \text{etc.}\end{aligned}\right\} \qquad (32A)$$

in which the coefficients are complicated quantities,—the numerators being computed by the expression

$$\triangle^n 0^m = n^m - n(n-1)^m + \frac{n(n-1)(n-2)^m}{1.2} - \frac{n(n-1)(n-2)(n-3)^m}{1.2.3} + \text{etc.} \qquad (32B)$$

I understand that in some editions of Boole's work the numerators have been computed and tabulated. Applied to the differentials these formulae become:

* Macmillan & Co., London, 1872.

$$\left.\begin{aligned}
\triangle_1 &= \frac{du}{dx} + \frac{1}{2}\frac{d^2u}{dx^2} + \frac{1}{6}\frac{d^3u}{dx^3} + \frac{1}{24}\frac{d^4u}{dx^4} + \frac{1}{120}\frac{d^5u}{dx^5} \\
\triangle_2 &= \frac{d^2u}{dx^2} + \frac{d^3u}{dx^3} + \frac{7}{12}\frac{d^4u}{dx^4} + \frac{1}{4}\frac{d^5u}{dx^5} \\
\triangle_3 &= \frac{d^3u}{dx^3} + \frac{3}{2}\frac{d^4u}{dx^4} + \frac{5}{4}\frac{d^5u}{dx^5} \\
\triangle_4 &= \frac{d^4u}{dx^4} + 2\frac{d^5u}{dx^5} \\
\triangle_5 &= \frac{d^5u}{dx^5}
\end{aligned}\right\} \quad (33)$$

Inserting the values of the differential coefficients from Equation (30)

$$\left.\begin{aligned}
\triangle_1 &= 405 + 270 + 90 + 15 + 1 = 781 \\
\triangle_2 &= 540 + 540 + 210 + 30 = 1320 \\
\triangle_3 &= 540 + 540 + 150 + = 1230 \\
\triangle_4 &= 360 + 240 = 600 \\
\triangle_5 &= 120 = 120
\end{aligned}\right\} \quad (34)$$

These differences agree with those in Example 6 lying on the diagonal line ending at F.

26. According to Boole the equation for the several differences in terms of the Differential coefficients, equation (32) is the fundamental relation between Differences and Differentials. From equation (33) it is again seen that the first differential coefficient is not the same as the First Difference which is perhaps the chief fallacy in the usual explanation of the calculus.

The only exception to this is that the *n*th Differential coefficient is always equal to the *n*th Difference which is a constant in both finite difference and differentials.

27. *Inversion of the Series*, in Example 6, Article 20.

It will be interesting to note what effect this inversion has upon the results of the two formulae in Articles 19 and 25. This is equivalent to taking equation 27 with the negative sign.

The Tabular differences then are as follows:

x	u	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\triangle_4$	$\triangle_5$	
4	+1024		+1320		+480		
		−781		−750		−120	
3	**243**		**570**		**360**		**D′**
		211		390		120	
2	32		**180**		240		
		31		**150**		120	
1	1		30		**+120**		
		− 1		− 30		**−120**	
0	+ 0		+ 0		+ 0		**F′**

The Differential Coefficients still lie in the horizontal line 3 — D′ while the Finite Differences are again in the downward diagonal 3 — F′ which now includes quite a different set of numbers from the previous table. Finite differences thus change when a series is inverted. Changes of signs are also noted above.

By formula 26 we find the Differentials from the above Differences as follows:

$$\left.\begin{array}{lll}
\frac{du}{dx} = -\frac{992}{2} + \frac{1}{6}\cdot\frac{1140}{2} - \frac{1}{30}\,120 = -496 + 95 - 4 = -405 \\
\frac{d^2u}{dx^2} = -570 - 30 + 540 \\
\frac{d^3u}{dx^3} = -\frac{1140}{2} + 30 = -570 + 30 = -540 \\
\frac{d^4u}{dx^4} = +360 = +360 \\
\frac{d^5u}{dx^5} = -120 = -120
\end{array}\right\} \quad (35)$$

Agreeing numerically with those previously found in the same manner and with the change of signs the inversion of the series necessitates, and also lying in the horizontal line of the tabulated differences.

By formula 33 we have the Finite Differences from the Differential Coefficients, employing those just found.

$$\left.\begin{array}{lll}
\triangle_1 = -405 + 270 - 90 + 15 - 1 = -496 + 285 = -211 \\
\triangle_2 = +540 - 540 + 210 - 30 = +750 - 570 = +180 \\
\triangle_3 = -540 + 540 - 150 = -150 \\
\triangle_4 = +360 - 240 = +120 \\
\triangle_5 = -120 = -120
\end{array}\right\} \quad (36)$$

Corresponding exactly with the new values of Finite Differences, lying in the diagonal line 3-F'. These two diagonal lines of Finite Differences are symmetrically disposed above and below the horizontal line of Differential Coefficients.

28. *Concluding Remarks on the Method by Limits.* The foregoing pages following example 6 Articles 19-20 show that the differential coefficients depend upon the interval of the argument x of finite differences being taken as unity; whatever the actual interval may be. We may, however, decrease the *actual* interval (Art. 15) while still regarding it as a unit, or even consider it infinitely small as customary in investigations. The author does not wish to be considered as holding that a differential must be unity or a large quantity. His object being only to prove that h in equations 2, 3, 4, 5 must be a unit and cannot be considered as a variable decreasing without limit.

Finally, the author asks if the method by limits can show the second differential coefficient? Can it be derived by another infinitely small quantity k? Does its derivation follow the same law as the first? It would be illogical to infer that it does, for that is begging the question—the very point we wish to know.

The method by Limits regards a differential coefficient as a *ratio* to which objection has been made; but finite differences being formed by subtraction, the differential coefficient derived from them is not a ratio.

RELATIVE MASSES OF BINARY STARS. II.

STEPHEN MARSHALL HADLEY.

IV.

DETERMINATION OF THE RELATIVE MASSES IN THE SYSTEM δ EQUULEI.

$\alpha = 21^h\ 9.6^m \qquad \delta = +9^\circ\ 37'$

This is a close binary having a period of 5.7 years according to Hussey.* It was first observed by Otto Struve in 1852. The plane of the orbit is highly inclined to the line of sight and the orbit is so placed that the change in right ascension due to the orbital motion does not exceed 0″.25. The components *A* and *B* are of about the same magnitude, 4.5 and 5 respectively. The neighboring star *C* is of the 9.2 magnitude. The position of *C* relative to $\frac{AB}{2}$ was first measured by William Struve in 1829, and it has been pretty well observed at subsequent dates as shown in Table IV. Between the dates 1869.67 and 1880.60 an extensive gap occurs, covering a period of almost two revolutions of the binary, in which no observations of *C* were found. In determining the five unknowns of equations (3) for this star the following observations have been used, in which the corrections have been applied for reducing the distances, ρ, and the position angles, θ, from $\frac{AB}{2}$ to *A* as origin, and for the reduction of the position angles, θ, to the equinox of 1900.

TABLE IV.

Adopted Coördinates *A*, *C*.

Date.	Nights.	θ ° ′	ρ ″	Weight.	Observer.	Reference.
1828.80	2	41 1	26.80	1	Σ	*c*
32.10	4	39 38	27.48	1	"	"
34.90	2	37 53	27.73	1	"	"
35.64	4	37 43	27.74	1	"	"
36.65	3	37 17	28.22	1	"	*f*
37.77	2	36 39	28.28	1	"	*c*
41.65	1	34 43	29.00	1	0Σ	*bl*
47.82	1	32 8	30.64	1	"	"
51.84	1	30 54	31.22	1	"	"
52.64	1	30 52	31.54	1	"	"
53.91	1	29 13	31.68	1	"	"
54.69	1	29 57	31.65	1	"	"
56.58	1	29 27	32.44	1	"	"
57.67	1	28 45	32.76	1	"	"
58.59	1	28 23	33.03	1	"	"
59.65	1	28 9	33.06	1	"	"
62.71	4	27 15	33.67	1	△	*i*
63.64	4	26 52	34.08	1	"	"
64.90	5	26 54	34.47	1	"	"

* Pub. *A. S. P.* Vol. XII.

Date.	Nights.	θ	ρ	Weight.	Observer.	Reference.
		° ′	″			
65.91	1	26 6	34.79	1	OΣ	*bl*
66.52	2	25 37	34.72	1	W	*a*
69.67	2	25 27	35.99	1	Du	*g*
80.60	5	22 41	38.15	1	β	*f*
81.46	3	22 17	38.95	1	"	"
83.65	3	22 2	39.26	1	Pe	*m* 107
87.71	2	21 9	40.28	1	HΣ	*d*
93.68	2	19 0	41.74	1	Gl	*e*
98.97	2	19 29	43.20	1	S	*m* 109
1902.71	3	27 35	43.86	1	C	*o*

The data, p and s, for the component B referred to A, were obtained from Hussey's* orbit for the dates of the available observations of C.

Observation equations were formed from these data, and solved by the method of least squares. This solution yielded the following elimination equations;

In Declination.	In Right Ascension.
$x + 0.000\,y + 0.256\,z + 0.014 = 0$	$x' + 0.000\,y' + 0.104\,z' - 0.004 = 0$
$y + 0.010\,z - 0.129 = 0$	$y' + 0.003\,z' - 0.340 = 0$
$z + 0.329 = 0$	$z' + 0.222 = 0$

in which

$$x = \Delta\alpha,\quad y = 41.916\,\Delta\beta,\quad z = \triangle k,\quad x' = \Delta\mu,\quad y' = 41.916\,\Delta\nu,\quad z' = \triangle k.$$

Their solution yields

$$\triangle\alpha = +0.070,\quad \triangle\beta = +0.0032,\quad \triangle k = -0.329$$
$$\triangle\mu = +0.027,\quad \triangle\nu = +0.0081,\quad \triangle k = -0.222$$

The assumed values were

$$\alpha_0 = +29.498,\quad \beta_0 = +0.2917,\quad k_0 = +0.50$$
$$\mu_0 = +15.548,\quad \nu_0 = -0.0580,$$

Therefore

$$\alpha = +29''.568 \pm 0''.038 \qquad \mu = +15''.575 \pm 0''.029$$
$$\beta = +\ 0''.2949 \pm 0''.0100 \qquad \nu = -\ 0''.0499 \pm 0''.0161$$
$$k = +\ 0\ .172 \pm 0\ .988 \qquad k = +\ 0\ .279 \pm 1.378$$

I find for the probable error of an equation of unit weight

In Declination $r_1 = \pm 0.165$. In Right Ascension $r_1 = \pm 0.267$.

The weights of the various quantities are as follows:

$$P_x = 4.35,\ P_y = 6.57,\ P_z = 0.334,\ P_{x'} = 9.32,\ P_{y'} = 6.57,\ P_{z'} = 0.150,$$
$$P_\alpha = 4.35,\ P_\beta = 275.22,\ P_k = 0.334,\ P_\mu = 9.32,\ P_\nu = 275.22,\ P_k = 0.150,$$

I then found the probable errors of α, β, μ, ν, and k as given above. The large probable errors in the values of k show that the data now available are wholly insufficient for determining the relative masses of the components of the binary δ Equulei. On the other hand the probable errors in the values of α, β, μ and ν indicate that these results are reliable. The values of α and μ given above are for the epoch 1860.79.

* Pub. *A. S. P.* Vol. XII.

V.

Determination of the Relative Masses in the System τ Cygni.

$\alpha = 21^h\ 10^m \qquad \delta = +37°\ 32''$

This was observed as a double star by Otto Struve in 1851. Several years later Alvan G. Clark discovered that the brighter of the two stars was a binary. It is known as A. G. C. 13. The magnitudes of the two components and of the neighboring star are respectively 4.2, 8.0 and 8.9. The binary has been well observed since the date of its discovery and the orbit seems to be well determined. The computed periods vary from 36.5 years to 57.25 years. I have obtained the data of the problem for A, B from Aitken's* orbit for the dates of the available observations of C and have used his period of 45.1 years in determining the position of B with respect to A at these dates. According to Aitken the length of the axis major of the apparent orbit is 1″.96 and of the axis minor 1″.20.

I have found only seven observations of A, C, six by Otto Struve between 1851 and 1883 and an unpublished one by Professor Comstock in 1903, as shown in the table below. I find no observation of A, C, when the position angle of B was between 0° and 90°, one when it was between 90° and 180°, two between 180° and 270° and four between 270° and 360°. This gives but one positive coördinate in right ascension and six negative, while in declination there are four positive coördinates and three negative.

Table V.

Adopted Coördinates A, C.

Date.	Nights.	θ °	′	ρ ″	Weight.	Observer.	Reference.
1851.84	1	213	1	136.53	1	0Σ	*b*2
52.66	1	212	42	135.88	1	"	"
52.69	1	212	47	136.85	1	"	"
54.78	1	212	32	138.21	1	"	"
66.72	1	212	00	142.98	1	"	"
83.87	1	211	19	151.15	1	"	"
1903.52	3	210	56	160.15	1	C	*o*

The data thus obtained have been used for obtaining the five unknowns in equation (3), the position angles, θ, are reduced to the equinox of 1900. The solution of the observation equations thus formed furnish the following elimination equations:

In Declination.

$$x + 0.000\,y + 0.109\,z - 0.011 = 0$$
$$y - 0.590\,z + 0.010 = 0$$
$$z + 0.332 = 0$$

In Right Ascension.

$$x' + 0.000\,y' - 0.346\,z' - 0.441 = 0$$
$$y' + 0.227\,z' + 0.401 = 0$$
$$z' + 0.069 = 0$$

in which

*Pub. *A. S. P.* Vol. XII.

$$x = \triangle\alpha, \qquad y = 36.94\ \triangle\beta, \qquad z = \triangle k$$
$$x' = \triangle\mu', \qquad y' = 36.94\ \triangle\nu, \qquad z' = \triangle k$$

Their solution yields

$$\triangle\alpha = +0.048, \qquad \triangle\beta = -0.0053, \qquad \triangle k = -0.332,$$
$$\triangle\mu = +0.417, \qquad \triangle\nu = -0.0104, \qquad \triangle k = -0.070,$$

The assumed values were

$$\alpha_0 = -121.219, \qquad \beta_0 = -0.4361, \qquad k_0 = +0.50$$
$$\mu_0 = -76.397, \qquad \nu_0 = -0.1514,$$

Therefore

$$\alpha = -121''.171 \pm 0''.159 \qquad \mu = -75''.980 \pm 0''.231$$
$$\beta = -0''.4414 \pm 0''.0664 \qquad \nu = -0''.1618 \pm 0''.0574$$
$$k = +0.168 \pm 0.359 \qquad k = +0.430 \pm 0.450$$

I find for the probable error of a unit weight

In Declination $r_1 = \pm 0.41$. In Right Asceesion $r_1 = \pm 0.46$

The weights of the various quantities are as follows:

$P_x = 6.637$, $P_y = 1.035$, $P_z = 1.362$, $P_{x'} = 3.982$, $P_{y'} = 1.747$, $P_{z'} = 1.093$,
$P_\alpha = 6.637$, $P_\beta = 38.233$, $P_k = 1.362$, $P_\mu = 3.982$, $P_\nu = 64.568$, $P_k = 1.093$.

The large probable errors in the values of k show that the data at present available are insufficient for determining the relative masses of A and B τ Cygni. More extensive and well distributed observations will probably furnish sufficient data for the solution of this problem.

The values of α and μ found above are for the epoch 1866.58.

VI.

Determination of the Relative Masses in the System κ Pegasi.

$$\alpha = 21^h\ 39^m \qquad \delta = +25^\circ\ 6'$$

The close pair was detected by Burnham in 1880. Various computors have found periods between 11 and 12 years. The component B has described about two revolutions since its discovery and has been well observed during that time. It is always a close pair, the maximum distance being less than 0″.30. A and B are of about the same magnitude 4.5 and 5 respectively while C is of about the tenth magnitude. Sir William Herschel discovered the wide pair but it was first measured by William Struve in 1831. The observation of $\frac{AB}{2}$ and C used in this discussion are set forth in Table VI. The coördinates, p and s, for the component B with respect to A, for the dates corresponding to the observation of C, were derived from Burnham's* orbit. The position angles, θ, have been re luced t › the equinox of 1900.

* Pub. Yerkes Observatory, Vol, I.

TABLE VI.

Observed Coördinates *A*, *C*.

Date.	Nights.	θ °	′	ρ ″	Weight.	Observer.	Reference.
1831.56	5	307	57	11.07	1	Σ	1
64.87	6	303	37	11.66	1	△	"
80.60	4	302	58	11.66	1	β	"
86.75	2	300	45	11.88	1	HΣ	*d*
88.81	3	299	44	12.18	1	Hl	1
88.82	2	300	26	12.27	1	β	"
88.92	2	300	32	12.00	1	Sp	"
93.66	2	297	29	11.71	1	Gl	"
94.82	2	298	50	12.05	1	L	*j*56
95.60	8	301	14	12.18	1	Ba	1
95.69	2	299	01	12.16	1	L	*j*56
96.70	1	297	48	12.13	1	"	*j*59
97.86	1	297	42	12.07	1	"	"
98.43	1	299	24	12.54	1	A	*n*150
98.71	3	299	38	12.56	1	L	*j*59
99.79	3	297	53	12.52	1	"	*j*60
1903.54	3	298	41	12.58	1	C	*o*

Observation equations were formed from these data and solved by the method of least squares. The solution yields the following elimination equations:

In Declination.

$$x + 0.000\,y - 0.025\,z + 0.161 = 0$$
$$y - 0.019\,z + 0.227 = 0$$
$$z + 0.250 = 0$$

In Right Ascension.

$$x' + 0.000\,y' - 0.023\,z' - 0.179 = 0$$
$$y' + 0.185\,z' - 0.012 = 0$$
$$z' - 0.169 = 0$$

in which

$$x = \Delta\alpha, \quad y = 56.977\,\Delta\beta, \quad z = \Delta k,$$
$$x' = \Delta\mu, \quad y' = 56.977\,\Delta\nu, \quad z' = \Delta k.$$

Their solution yields

$$\Delta\alpha = -\ 0.167, \quad \Delta\beta = -\ 0.0040, \quad \Delta k = -\ 0.250,$$
$$\Delta\mu = +\ 0.183, \quad \Delta\nu = -\ 0.0054, \quad \Delta k = +\ 0.169.$$

The assumed values were

$$\alpha_0 = +\ 6.205, \quad \beta_0 = -\ 0.0107, \quad k_0 = +\ 0.50$$
$$\mu_0 = -\ 10.609, \quad \nu_0 = -\ 0.0345,$$

Therefore,

$$\alpha = +\ 6''.038 \pm 0''.187 \qquad \mu = -\ 10''.426 \pm 0''.120$$
$$\beta = -\ 0''.0147 \pm 0''.0273 \qquad \nu = -\ 0''.0399 \pm 0''.0178$$
$$k = +\ 0\ .250 \pm 3\ .174 \qquad k = +\ 0\ .669 \pm 0\ .774$$

I find for the probable error of an equation of unit weight

In Declination $r_1 = \pm\ 0.25$. In Right Ascension $r_1 = \pm\ 0.16$

The weights of the various quantities are as follows:

$P_x = 1.680$, $P_y = 1.456$, $P_z = 0.084$, $P_{x'} = 1.697$, $P_{y'} = 1.345$, $P_{z'} = 0.561$,
$P_\alpha = 1.680$, $P_\beta = 82.943$, $P_k = 0.084$, $P_\mu = 1.697$, $P_\nu = 76.620$, $P_k = 0.561$.

The above probable errors in the values of α, β, μ, ν and k were then computed. The large probable errors in the values of k indicate that the data now available are wholly insufficient for the determination of the relative masses of components of κ Pegasi. The values of α and μ found above are for the epoch 1888. 54.

I have examined all the available records of micrometrical measurements of double star observations accessible to me and the above summary contains all the stars for which sufficient data

VII.

Summary of Results.

Star.	R.A.	Decl.	Magnitudes.			R.A. Coördinates.			Decl. Coördinates.		
	h m	° ′				k		Weight.	k		Weight.
σ Coronae	16 11	+34 7	4.5	5.0	10.2	+0.543	± 0.064	0.90	+0.502	± 0.068	0.20
β Delphini (A,B,C)	20 31.9	+14 11	4.3	5.9	12.7	−0.181	± 2.65	0.16	+0.113	± 1.04	0.27
" (A,B,D)			4.3	5.9	11.0	+0.070	± 1.54	0.44	+0.193	± 0.88	1.07
δ Equulei	21 9.6	+ 9 36	4.5	5.0	9.2	+0.279	± 1.38	0.15	+0.117	± 0.998	0.33
τ Cygni	21 10	+37 32	4.2	8.0	8.9	+0.430	± 0.450	1.09	+0.168	± 0.359	1.36
κ Pegasi	21 39	+25 6	4.5	5.0	10.0	+0.699	± 0.774	0.56	+0.250	± 3.174	0.08

Star.	Epoch.	Motion in R.A.				Motion in Decl.			
		μ		ν		α		β	
		"	"	"	"	"	"	"	"
σ Coronae	1864.73	+51.877	± 0.155	+0.2983	± 0.0347	+ 3.037	± 0.308	+0.0760	± 0.0100
β Delphini (A,B,C)	1889.37	+23.748	± 0.063	−0.1118	± 0.0255	− 11.913	± 0.181	+0.0158	± 0.0088
" (A,B,D)	1881.30	−32.512	± 0.161	−0.0327	± 0.0447	+ 15.272	± 0.193	+0.1119	± 0.0508
δ Equulei	1860.79	+15.575	± 0.029	−0.0499	± 0.0161	+ + 29.568	± 0.038	+0.2949	± 0.0100
τ Cygni	1866.58	−75.980	± 0.231	−0.1618	± 0.0574	−121.171	± 0.159	−0.4414	± 0.0664
κ Pegasi	1888.54	−10.426	± 0.120	−0.0399	± 0.0178	+ 6.038	± 0.187	−0.0147	± 0.0273

were found to justify an attempt to determine the ratio of the masses of the components of the binary by the method here employed. I have also examined the records with a view of find-

ing other binaries having optical companions for which additional measures of the binary and companion would probably furnish data for determining the relative masses of the components in the comparatively near future; but the search failed to bring any such to light.

The most pressing cases seemed to be the following for which additional observations are desired. The first column contains the dates of the measures given.

			θ	ρ	Mag.	Remarks.
1902.81	β Delphini	$\frac{AB}{2}$, D	331° 7′	36″.86	11	Once every one or two years.
1898.59	"	$\frac{AB}{2}$, C	117 6	25 .92	12	
1902.71	δ Equulei	$\frac{AB}{2}$, C	17 30	43 .74	9.2	Once each year or oftener.
1903.52	τ Cygni	A,C	210 55	160 .15	89.	Once in two or three years.

VIII.

Determination of Mass from Observations of Absolute Positions.

In a few cases it has been found feasible to determine from meridian circle observations the apparent orbit of one component of a binary star. When this has been done a very few micrometrical measurements of the two components of the binary will suffice to determine their relative masses. For the sake of completeness there is therefore appended an outline of the determination of such orbits from meridian circle observations following in substance the method of Peters and Auwers as set forth in the *Astronomische Nachrichten* Nos. 746, 747 and 1372.

Let the plane through the center of gravity, O, of the binary system and perpendicular to the line of sight be the plane xy, the positive direction of the x-axis being directed toward the north and of the y-axis toward the east.

Let δ = the declination of the star at the time t,

D = the declination of 0 at the time t,

α = the R. A. of the star at the time t,

A = the R. A. of 0 at the time t,

Put

$$\delta - D = \xi \text{ and } 15\,(\alpha - A) = \eta$$

then for the time t,

$$x = \xi \text{ and } y = \eta \cos \delta.$$

x and y are functions of the elements of the orbit, if u is the eccentric anomaly then

$$y = \eta \cos \delta = (f + g \cos u + h \sin u) \cos \delta$$
$$x = \xi \qquad = f' + g' \cos u + h' \sin u$$

where according to Peters *A. N.* No. 746

$$\begin{aligned} g &= a \sec \delta \,(\cos w \sin \Omega + \sin w \cos \Omega \cos i) \\ h &= -a \sec \delta \sqrt{1-e^2}\,(\sin w \sin \Omega - \cos w \cos \Omega \cos i) \\ f &= -ge \\ g' &= a\,(\cos w \cos \Omega - \sin w \sin \Omega \cos i) \\ h' &= -a\sqrt{1-e^2}\,(\sin w \cos \Omega + \cos w \sin \Omega \cos i) \\ f' &= -g'e \end{aligned}$$

Let δ^0 = the mean declination of the star at the time t^0,
$\delta_r^{\,0}$ = the mean declination of the star at the time t^0 computed from reduction tables.

Put

$$\delta^0 - \delta_r^{\,0} = k'$$

Then the declination of the center of gravity, O, at the time t^0 equals $\delta_r^{\,0} + k' - \xi^0$, or

$$D^0 = \delta_r^{\,0} + k' - (f' + g' \cos u + h' \sin u)$$

Let c' = the annual change in the declination of O derived from the annual change in the declination of the star,
δ_r = the declination of the star at the time t computed from reduction tables.

Then for the time t

$$D = \delta_r + k' + c'\,(t - t^0) - (f' + g' \cos u + h' \sin u)$$

Now

$$\delta = D + \xi$$

Hence

$$\delta = \delta_r + k' + c'\,(t - t^0) + g'\,(\cos u - \cos u_0) + h'\,(\sin u - \sin u_0)$$

Each observation will make known $\delta - \delta_r$, and will furnish an observation equation of the above type.
If the R. A. has been observed the corresponding equation is

$$\alpha = \alpha_r + k + c\,(t - t^0) + \frac{1}{15}\,g\,(\cos u - \cos u^0) + \frac{1}{15}\,h\,(\sin u - \sin u^0)$$

where α_r is the R. A. of the star computed from reduction tables.
If we place

$$\begin{aligned} k' - (g' \cos u^0 + h' \sin u^0) &= b' + f' \\ k - \frac{1}{15}\,(g \cos u^0 + h \sin u^0) &= b + f \\ \delta - \delta_r &= r \\ \alpha - \alpha_r &= q \end{aligned}$$

the above equations take the form

$$r = b' + f' + c'\,(t - t^0) + g' \cos u + h' \sin u \qquad (1)$$

$$q = b + f + c\,(t - t^0) + \frac{1}{15}\,g \cos u + h \sin u \qquad (2)$$

If T is the epoch of peri-astron and n the mean yearly motion of the star in its orbit the mean anomaly, at the time t, equals $n\,(t-T)$ and is connected with the eccentric anomaly by the relation

$$n\,(t-T) = u - e \sin u \qquad (3)$$

r is a function of the unknowns n, T, e, $(b' + f')$, c', g' *and* h'; q is a function of the unknowns, n, T, e, $(b + f)$, c, g and h.

Each observation in declination will furnish an equation of condition of the form (1), and each observation in right ascension will furnish an equation of condition of the form (2).

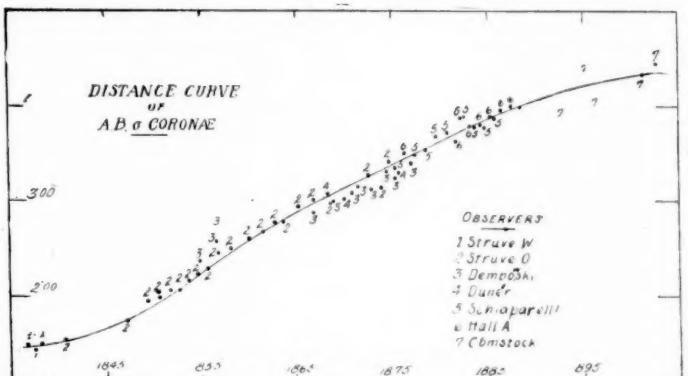
A set of equations of condition formed from a sufficient number of observations in declination or in right ascension, with equation (3) will yield values of the unknowns and thus lead to the elements of the orbit.

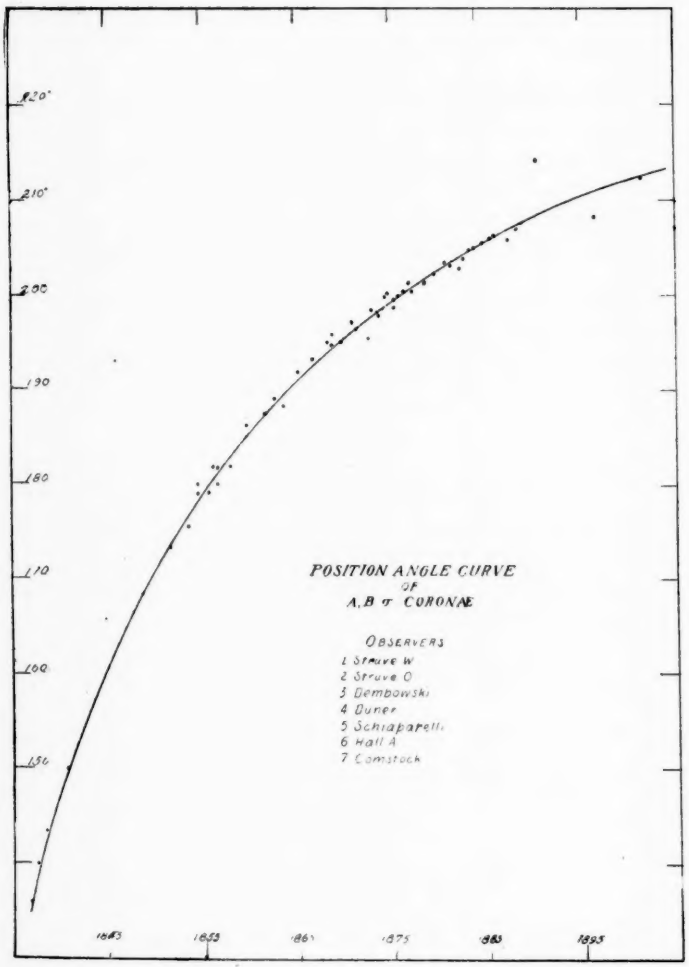
After the orbit of one of the components has been found, a few micrometrical measurements of the two components will suffice to make known the relative masses by the following simple formula.

$$\frac{a'}{s} = \frac{M_1}{M_1+M_2}$$

where M_1 and M_2 represent the masses of the two components of the binary, a^1 represents the radius vector of the known orbit at the time the observation is made and s the distance between the two components at that time.

(The following cuts should have appeared in the May Number, opposite page 263.)




220°
210°
200
190
180
170
160
150
1843
1853
1863
1875
1883
1893
POSITION ANGLE CURVE
OF
A, B σ CORONAE
OBSERVERS
1 Struve W
2 Struve O
3 Dembowski
4 Duner
5 Schiaparelli
6 Hall A
7 Comstock

DOUBLE STAR OBSERVATIONS AT THE CAPE OF GOOD HOPE.

S. W. BURNHAM.

FOR POPULAR ASTRONOMY.

Volume II of the *Annals of the Royal Observatory*, Cape of Good Hope, has recently been published, Part IV of which contains the measures of double stars made at that Observatory during the Directorship of Sir David Gill. The work is compiled by R. T. A. Innes, and the greater part of the measures were made by him in the years 1899—1903. These are all southern pairs, and most of them too far south for observatories in the northern hemisphere. For that reason observations are specially needed. Very little, comparatively speaking, has been done in the re-observation and measurement of the double stars catalogued by Herschel three-quarters of a century ago. Many important additions to the list of southern pairs have been made by discoveries at the Flagstaff, Cape and Australian observatories, and many of these objects should be watched, and measured at intervals. As far as practicable this matter seems to have received attention at the Cape Observatory. Unfortunately Mr. Innes' connection with it ceased in 1903, when he was appointed Director of the Transval Observatory at Johannesburg. That institution at present has no equatorial. It is to be hoped that his zeal and enthusiasm in this field of astronomical research will result in acquiring a suitable instrument for continuing the micrometrical work on the known pairs, and the further discovery of new ones. He is already credited with the addition of nearly 300 double stars not previously known.

Sir David Gill in the preface of the volume above referred to speaking of the routine character of the work at the Cape Observatory in previous years, says:

"Thus at the Cape Observatory, as has always been the case elsewhere, the subject of double star measurement on any great scale waited for the proper man to undertake it. There is no instance, so far as I know, of a long and valuable series of double star discovery and observation made by a mere assistant acting under orders. It is a special faculty, an inborn capacity, a delight in the exercise of exceptional acuteness of eyesight and natural dexterity, coupled with the gift of imagination as to the true meaning of what he observes, that imparts to the observer the requisite enthusiasm for double star observing. No amount of training or direction could have created the Struves, a Dawes

or a Dembowski. The great double star observer is born, not made, and I believe that no extensive series of double star discovery and measurement will ever emanate from a regular observatory through successive directorates unless men are specially selected who have previously distinguished themselves in that field of work, and who were originally driven to it from sheer compulsion of inborn taste."

There may not be very much that is specially new in this view of the subject matter, but no one heretofore has said these things so well. There is not now, and never has been, any lack of theorists and computors, who are far ahead of the facts, and of those who furnish them. Generally speaking orbits are not now wanted because the necessary data to give them any value, even provisionally, is altogether lacking. Measures, and particularly good measures by careful and experienced observers, are always needed. To many if not most of those who have in the past dealt with the theoretical and mathematical side of the subject, a measure is a measure, like the yellow primrose, and nothing else, and the final result of stirring these things together in the analytical crucible has generally turned out to be of very uncertain value, or no value at all. The best material is none too good, and it is certainly not improved by a mixture with crude work which any well-advised authority would unhesitatingly reject at sight. The advancement of our knowledge of the sideral systems depends wholly upon work with the micrometer. The measures of this year cannot be made ten years hence, but the orbit can be investigated then, and much better if the motion in the mean time is sufficient to warrant the attempt.

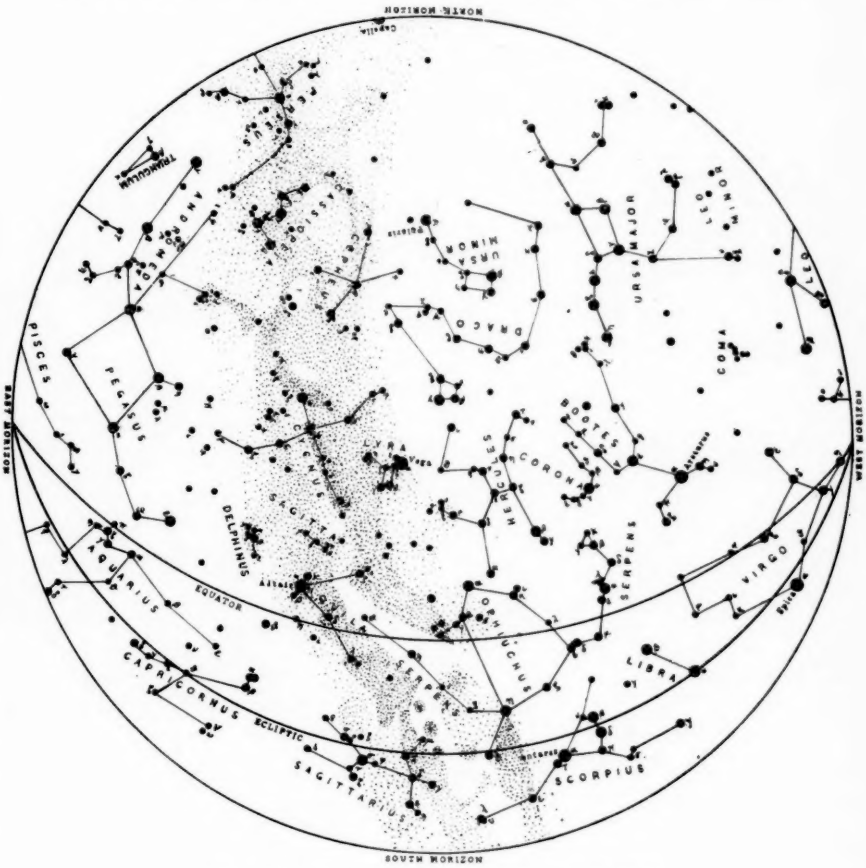
The mean results of most of the measures by Innes and Lunt have appeared in various numbers of *Monthly Notices* from time to time, but their convenient arrangement here in one volume will make them much more useful hereafter when they are needed for reference. It contains also a few measures made from 1849 to 1868 by Maclear, or under his direction. The work as a whole is carefully done, and free from accidental errors of all kinds.

CHICAGO, Ill.

PLANET NOTES FOR JULY AND AUGUST, 1905.

H. C. WILSON.

Mercury will be evening star during the greater part of these two months and may be seen with the naked eye during the last week of July and the first week of August. Its brightness will be comparatively low at this time, being



THE CONSTELLATIONS AT 9 P. M. AUGUST 1, 1905.

only half as bright as at the greatest eastern elongation on April 3. Mercury will be at greatest eastern elongation on the morning of Aug. 2, and will reach inferior conjunction Aug. 29.

Venus is morning star and will be at greatest elongation, west from the Sun,

45° 44′ on the morning of July 6. Her brilliancy will decrease in the ratio of 145 July 1 to 81 Aug. 31. The phase will increase in the same time from 0.470 to 0.740.

Mars will be in position for early evening observations during these months and it will be well for observers to study the planet in the twilight as well as in the dark of the night in order to see under what conditions the puzzling markings of Mars' surface can best be seen. The planet was at its brightest for this opposition in May and will diminish quite rapidly in July and August. The apparent diameter of the disk will diminish in the same time from 15″ to 12″. The planet's course will be southeastward from Libra into Scorpio. Mars will be at quadrature Aug 26.

Jupiter is morning star, not far from Venus, and may be studied in the morning twilight. The two brilliant planets will be in conjunction on the morning of July 4, Venus being then 2° 31′ south of Jupiter. Jupiter will be at quadrature 90° west from the Sun Aug. 28.

Saturn is in Aquarius and will be at opposition to the Sun August 22. The planet can therefore be observed in the middle of the night during these months. There are no very bright stars in that part of the sky so that Saturn is recognized at once by his brightness as well as by his golden color.

Uranus may be found in the evening among the faint stars in the Milky Way in Sagittarius. Its position July 15 is R.A. $18^h\ 06^m\ 06^s$, Decl. −23° 42′; and Aug. 15 R.A. $18^h\ 02^m\ 09^s$, Decl. −23° 43′.

Neptune having just passed conjunction June 29 is not in position for observation during the summer.

Occultations Visible at Washington.

Date. 1905.	Star's Name.	Magnitude.	IMMERSION. Washington M.T.	IMMERSION. Angle f'm N pt.	EMERSION. Washington M. T.	EMERSION. Angle f'm N pt.	Duration.
			h m	°	h m	°	h m
July 9	f^2 Virginis	4.8	10 02	122	11 05	277	1 03
14	Lalande 33327	6.3	5 57	154	6 34	225	0 37
15	B.A.C. 6616	6.4	12 45	54	13 57	288	1 12
17	29 Capricorni	5.5	13 46	45	14 54	290	1 08
22	33 Ceti	6.1	12 29	127	13 04	186	0 35
24	μ Ceti	4.3	14 02	70	15 14	243	1 12
25	f Tauri	4.3	13 06	85	14 05	237	0 59
26	γ Tauri	3.9	13 08	31	13 47	299	0 39
26	70 Tauri	6.4	15 52	53	17 0	270	1 08
Aug. 9	B.A.C. 5712	6.5	7 42	63	8 49	318	1 07
11	Lalande 35497	6.1	12 56	8	13 17	333	0 21
13	Mayer 889	5.7	15 17	94	16 17	229	1 00
14	42 Capricorni	5.1	9 53	122	10 50	204	0 57
15	σ Aquarii	4.8	9 15	91	10 30	232	1 15
17	4 Ceti	6.3	11 32	18	12 28	291	0 56
17	5 Ceti	6.3	12 06	2	12 46	305	0 40
19	ν Piscium	4.6	13 24	19	14 26	285	1 02
22	DM. +14°, 657	5.9	16 12	66	17 41	254	1 29
23	Bradley 686	5.7	15 36	49	16 48	279	1 12
26	f Geminorum	5.3	14 55	89	15 52	268	0 57

COMET AND ASTEROID NOTES.

Definitive Orbit of Comet 1903 III.—In *A. J.* No. 573 Mr. Henry A. Peck gives the following definitive elements of the bright comet, discovered April 17, 1903 by Grigg at Thames, New Zealand. These elements are subject to con-

siderable uncertainty owing to the small number and rough character of the observations which were available.

ELEMENTS OF COMET 1903 III.

T = March 25.42031 ± 0.00279 Gr. M. T.
ω = 184° 57′ 01″ ± 1′ 36″
Ω = 213 08 03 ± 8 } 1903.0
i = 66 29 36 ± 2
log q = 9.698466 ± 0.000257

Ephemeris of Comet 1904 I.

1905		α			δ		log Δ	H
		h	m	s	°	′		
June	1	8	3	1	+48	47.6		
	3		3	46	48	32.8	0.7630	0.041
	5		4	33	48	18.2		
	7		5	22	48	3.9	0.7683	0.040
	9		6	12	47	49.9		
	11		7	4	47	36.2	0.7733	0.039
	13		7	57	47	22.7		
	15		8	52	47	9.5	0.7780	0.038
	17		9	48	46	56.6		
	19		10	45	46	44.0	0.7824	0.037
	21		11	43	46	31.7		
	23		12	41	46	19.6	0.7866	00.36
	25		13	40	46	7.8		
	27		14	40	45	56.2	0.7905	0.035
	29		15	41	45	44.9		
July	1		16	42	45	33.9	0.7941	0.034
	3		17	44	45	23.1		
	5		18	46	45	12.6	0.7974	0.033
	7		19	48	45	2.3		
	9		20	50	44	52.3	0.8004	0.032
	11		21	53	44	42.5		
	13		22	56	44	33.0	0.8031	0.031
	15		23	59	44	23.8		
	17		25	2	44	14.8	0.8056	0.031
	19		26	5	44	6.0		
	21		27	7	43	57.4	0.8078	0.030
	23		28	9	43	49.1		
	25		29	11	43	41.1	0.8097	0.029
	27		30	13	43	33.3		
	29		31	14	43	25.8	0.8114	0.029
	31		32	15	43	18.5		
Aug.	2		33	15	43	11.5	0.8128	0.028
	4		34	15	43	4.7		
	6		35	14	42	58.1	0.8139	0.028
	8		36	12	42	51.8		
	10		37	9	42	45.7	0.8147	0.028
	12		38	5	42	39.9		
	14		39	0	42	34.3	0.8153	0.027
	16		39	54	42	29.0		
	18		40	48	42	23.9	0.8157	0.027
	20		41	40	42	19.1		
	22		42	31	42	14.6	0.8158	0.027
	24		43	21	42	10.3		
	26		44	9	42	6.3	0.8156	0.026
	28		44	56	42	2.5		
	30		45	42	41	59.0	0.8152	0.026
Sept.	1		46	26	41	55.7		
	3	8	47	9	+41	52.7	0.8145	0.026

Ephemeris of Comet 1904 II (1904 *d*).

1905		α vera			δ vera			log *r*	log Δ	H
		h	m	s	°	′				
June	1	2	35	42	+64	50.8		0.4945	0 5709	0.16
	3		39	27	64	51.6				
	5		43	.5	64	52.6		9.4994	0.5748	0.15
	7		46	37	64	53.7				
	9		50	3	64	55.1		0.5041	0.5782	0.15
	11		53	23	64	56.7				
	13		56	36	64	58.5		0.5088	0.5814	0.14
	15	2	59	43	65	0.5				
	17	3	2	44	65	2.8		0.5135	0.5841	0.14
	19		5	39	65	5.3				
	21		8	28	05	8.1		0.5182	0.5865	0.13
	23		11	11	65	11.1				
	25		13	48	65	14.3		0.5228	0.5886	0.13
	27		16	19	65	17.8				
	29		18	43	65	21.5		0.5274	0.5903	0.12
July	1		21	1	65	25.4				
	3		23	13	65	29.5		0.5319	0.5917	0.12
	5		25	18	65	33.9				
	7		27	17	65	38.5		0.5364	0.5928	0.12
	9		29	9	65	43.2				
	11	3	30	53	+65	48.2		0.5408	0.5936	0.12

VARIABLE STARS.

Three New Variables, 44, 45 and 46.1905.—These are announced by Prof. Ceraski in *A. N.* 4010 and were discovered by Mme. L. Ceraski on photographic plates taken by M. S. Blajko at Moscow. Their positions are as follows:

	R.A. 1855.	Decl. 1855.	R.A. 1900.	Decl. 1900.
	h m s	° ′ ″	h m s	° ′ ″
44. 1905 Andromedæ	23 13 44	+38 50.8	23 15 53	+39 05.4
45. 1905 Tauri	4 38 11.82	+25 54 43.2	4 40 57.58	+25 59 53.4
46. 1905 Cassiopeiae	1 27 48.6	+57 00.9	1 30 42.2	+57 14.8

The first of these stars was invisible on plates taken Oct. 3, 1900, Aug. 19, 1901, Oct. 1, 1902 and Aug. 15 and Sept. 6 1904, but was about 10th magnitude on a photograph taken Feb. 3, 1905.

The second star 45. 1905 is BD. + 25° 732. The estimates of its brightness are obtained from the plates:

Date	Mag.		Date	Mag.
Dec. 16 1895	9.3	Jan.	23 1905	9.3
21 "	9.3		26 "	9.1
Mar. 25 1897	9.5	Feb.	23 "	10.4
Jan. 16 1898	10.3		24 "	10.1
Mar. 18 1900	<10.0		25 "	10.5
29 "	10.2			

On March 14, 15 and 16 Mr. Blajko noted the magnitude as 8.5 and the color as reddish.

The third variable is BD. + 57° 342 and was estimated at 9.0 magnitude by Mr. Blajko on March 17. An examination of the Moscow photographs furnishes the following data:

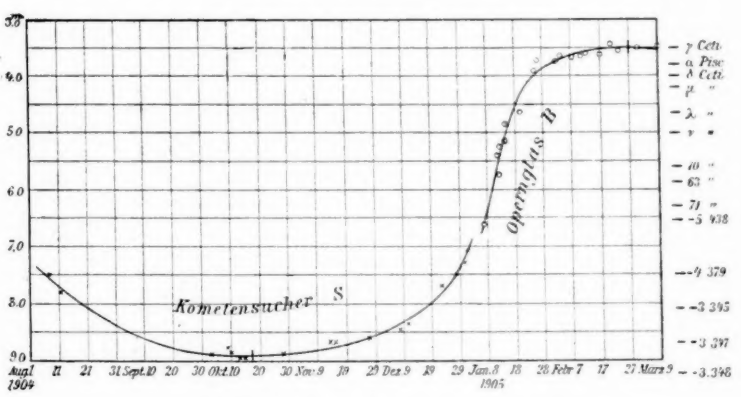
	Date.	Mag.		Date.	Mag.
Nov.	23 1895	9.1	Jan.	17 1900	9.6
	25 "	9.1		18 "	9.7
Oct.	4 1896	9.4	Aug.	29 "	10.0
	12 "	11.0	Apr.	17 1903	9.3
Mar.	24 1898	9.8	Jan.	11 1904	10.5
Sept.	24 "	10.2	Aug.	12 "	9.7
Nov.	10 "	8.9	Sept.	13 "	9.0
Nov.	30 1899	9.0	Mar.	1 1905	10.4

New Variable 47.1905 Aurigae.—In *A. N.* 4010 Prof. H. H. Turner states that "Mr. T. H. Astbury of Craft Villas, Wallingford, has found the star 48 Aurigae ($6^h\ 19^m\ 14.^s9 + 30°\ 34.'7$ 1855) to be variable. The range is about half a magnitude, from about 5.0 to 5.5, and the period is probably a few days only."

New Variable Star 59.1905 Lyrae.—In *A. N.* 4013 Messrs. M. and G. Wolf, of Heidelberg announce what may possibly be a new star of the tenth magnitude, found upon two photographs taken April 13, 1905 and not found on earlier photographs. Its position for 1905.0 is

$$RA = 18^h\ 50^m\ 17.^s07;\quad Decl. = +\ 36°\ 23'\ 23.''9$$

The Variable Star Mira (o Ceti).—In *A. N.* 4012 and 4013 Prof. A. A. Nijland of Utrecht gives the results of his observations of *Mira* from July 13,



THE MIRA MINIMUM OF OCTOBER 1904.

1903 to Feb. 23, 1904 and from Aug. 8, 1904 to March 10, 1905 The minima were about the middle of November 1903 and the middle of October 1904. On February 23, 1904 the star had almost reached its maximum and on March 10, 1905 the maximum was apparently past.

Maxima and Minima of Long Period Variables.—In *A. J.* 573 Mr. J. A. Parkhurst gives the resulting maxima and minima of the following long period variable stars from his series of observations during the years 1893-1895: *T* Andromedæ, *V* Andromedæ, *W* Andromedæ *R U* Herculis, *R V* Herculis, *S* Lyræ,

S Cygni, *SX* Cygni, *V* Delphini, *Y* Cassiopeiæ. Mr. Parkhurst finds the period of *S* Lyræ to be 438 days, instead of 218 as given by Chandler, and that of *SX* Cygni to be 409 instead of 548 days.

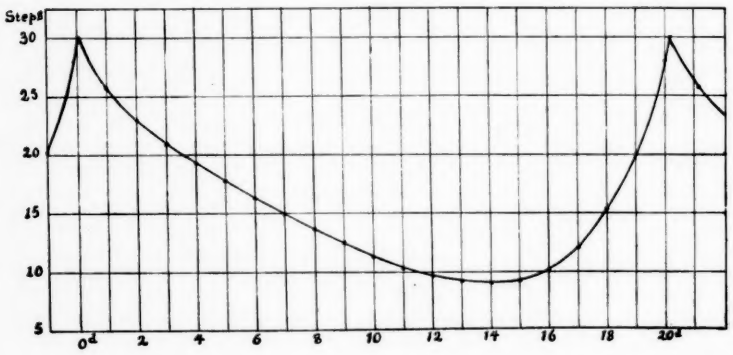
In the same number of the *Astronomical Journal* Mr. A. Stanley Williams gives the results of his observations of *RU* Andromedæ, *V* Andromedæ, *RV* Andromedæ, *Y* Persei, *RY* Lyræ, *RW* Lyræ, *RX* Lyræ, *Z* Lyræ, *RT* Lyræ, *RU* Lyræ, *TY* Cygni, *UX* Cygni and *TW* Cygni. Mr. Williams gives the following elements of two of these long period variables:

RW Lyræ = Maximum 1900 Oct. 6 (J. D. 2415299) + 496^{d} E
RT Lyræ = Maximum 1902 July 22 (J. D. 2415953) + 248.^{d}7 E.

The Variable Star VX Cygni.—In *A. N.* 4010 Mr. A. Stanley Williams, from eighteen maxima observed in the years 1899-1904, determines the following elements.

Maximum = J. D. 2414934.97 + 20^{d}.125 E.

The maximum brightness of the variable is 9.07^{m} and the minimum 10.33^{m}.



THE LIGHT CURVE OF VX Cygni.

The maximum is a very sharply accentuated one, the last half of the rise and the first part of the fall being very rapid. Both the visual and photographic observations suggest that the decline, after some maxima, takes place more quickly than it does after others. The interval from minimum to maximum is 6.2 days.

Mr. Williams gives the following list of comparison stars:

		RA. 1855		Decl.		Light Scale.	Mag.
		h	m	°			
c BD. + 39 4381 (8.8)		20	52.0	39	29	36.2	8.70
d BD. + 39 4370 (9.0)		20	50.1	39	34	28.8	9.14
e		20	51.2	39	37	12.1	10.14
f		20	52.1	39	40	0.0	10.87

Observation of Variable Stars.—In *A. N.* 4011 Dr. L. Terkáss gives the results of his observations during the years 1903 and 1904 of the short period variables S Sagittae, T Vulpeculae, S Cephei, η Aquilae, β Persei and λ Tauri. For the first four he gives light curves.

Variable Stars of Short Period not of the Algol Type.

	Minimum.		Maximum.			Minimum.		Maximum.	
	d	h	d	h		d	h	d	h
S Crucis	July 1	0	July 2	12	R Triang. Austr.	July 13	19	July 14	19
SU Cygni	1	4	2	12	R Crucis	14	10	15	19
η Aquilae	1	6	3	15	S Crucis	15	3	16	15
δ Cephei	1	18	3	3	T Vulpeculae	15	11	16	20
VY Cygni	1	18	3	20	η Aquilae	15	14	17	23
T Vulpeculae	2	3	3	12	Y Ophiuchi	15	23	22	6
V Centauri	2	8	3	19	κ Pavonis	16	2	20	11
X Sagittarii	2	11	5	8	Y Sagittarii	16	8	18	10
R Crucis	2	18	4	3	RV Scorpii	16	8	17	17
U Sagittarii	2	19	5	18	X Sagittarii	16	11	19	8
T Velorum	3	4	4	13	SU Cygni	16	12	17	20
U Aquilæ	3	11	5	15	U Sagittarii	16	13	19	12
R Triang. Austr.	3	15	4	15	S Normae	16	20	21	5
β Lyræ	3	23	7	1	β Lyræ	16	21	19	23
U Vulpeculae	4	1	6	4	S Muscae	17	1	20	12
RV Scorpii	4	5	5	14	T Velorum	17	2	18	11
V Velorum	4	7	5	6	R Triang. Austr.	17	5	18	5
Y Sagittarii	4	19	6	21	V Velorum	17	10	18	9
SU Cygni	5	0	6	8	VY Cygni	17	11	19	13
S Triang. Austr.	5	2	7	4	U Aquilae	17	11	19	15
T Crucis	5	11	7	12	S Sagittae	17	16	21	2
S Crucis	5	17	7	5	S Triang. Austr.	17	17	19	19
V Carinæ	6	4	8	8	δ Cephei	17	20	19	5
W Sagittarii	6	4	9	4	V Centauri	18	20	20	7
X Cygni	6	11	11	14	T Crucis	18	21	20	22
T Vulpeculae	6	14	7	23	V Carinae	19	4	21	8
R Triang. Austr.	7	0	8	0	S Crucis	19	19	21	7
κ Pavonis	7	0	11	9	T Vulpeculae	19	21	21	6
S Normae	7	2	11	11	U Vulpeculae	20	0	22	3
δ Cephei	7	3	8	12	R Crucis	20	5	21	14
S Muscae	7	9	10	20	SU Cygni	20	9	21	17
T Velorum	7	19	9	4	R Triang. Austr.	20	14	21	14
V Centauri	7	20	9	7	W Sagittarii	21	8	24	8
W Virginis	8	2	16	6	T Velorum	21	17	23	2
η Aquilae	8	10	10	19	V Velorum	21	19	22	18
R Crucis	8	14	9	23	Y Sagittarii	22	3	24	5
V Velorum	8	16	9	15	RV Scorpii	22	9	23	18
SU Cygni	8	20	10	4	η Aquilae	22	18	25	3
S Sagittae	9	7	12	17	X Cygni	22	21	28	0
X Sagittarii	9	11	12	8	U Sagittarii	23	1	26	0
U Sagittarii	9	13	12	12	δ Cephei	23	5	24	14
VY Cygni	9	15	11	17	β Lyræ	23	8	26	15
RV Scorpii	10	6	11	15	X Sagittarii	23	11	26	8
R Triang. Austr.	10	10	11	10	R Triang. Austr.	23	23	24	23
S Crucis	10	10	11	22	S Triang. Austr.	24	1	26	3
β Lyrae	10	10	13	17	SU Cygni	24	5	25	13
U Aquilae	10	11	12	15	T Vulpeculæ	24	8	25	17
Y Sagittarii	10	14	12	16	V Centauri	24	8	25	19
T Vulpeculæ	11	0	12	9	S Crucis	24	12	26	0
S Triang. Austr.	11	9	13	11	U Aquilae	24	12	26	16
TX Cygni	11	12	16	15	κ Pavonis	25	4	29	13
U Vulpeculae	12	1	14	4	VY Cygni	25	8	27	10
T Crucis	12	4	14	5	W Virginis	25	9	34	1
T Velorum	12	10	13	19	T Crucis	25	16	27	17
δ Cephei	12	12	13	21	R Crucis	26	1	27	10
SU Cygni	12	16	14	0	S Sagittae	26	1	29	11
V Carinae	12	20	15	0	V Velorum	26	4	27	3
V Velorum	13	1	14	0	TX Cygni	26	5	31	8
V Centauri	13	8	14	19	V Carinæ	26	6	28	10
W Sagittarii	13	18	16	18	T Velorum	26	9	27	18

Variable Stars of Short Period not of the Algol Type.—Continued.

	Minimum.		Maximum.	
	d	h	d	h
S Normae July	26	14	July 30	23
S Muscae	26	16	30	3
T Monocerotis	27	1	33	23
R Triang. Austr.	27	8	28	8
Y Sagittarii	27	21	29	23
U Vulpeculae	28	0	30	3
SU Cygni	28	2	29	11
RV Scorpii	28	11	29	20
δ Cephei	28	14	29	23
T Vulpeculae	28	18	30	3
S Crucis	28	18	30	6
W Sagittarii	28	23	31	23
U Sagittarii	29	19	32	18
β Lyræ	29	19	32	21
V Centauri	29	20	31	7
η Aquilae	29	22	32	7
S Triang. Austr.	30	9	32	11
X Sagittarii	30	11	33	8
V Velorum	30	12	31	11
R Triang. Austr.	30	18	31	18
T Velorum	31	0	Aug. 1	9
U Aquilae	31	12	2	16
R Crucis	31	21	2	6
SU Cygni	31	22	2	6
T Crucis Aug.	1	9	3	10
V Carinae	1	22	4	2
Y Ophiuchi	2	1	8	6
VY Cygni	2	4	4	6
T Vulpeculae	2	5	3	14
S Crucis	2	11	3	23
Y Sagittarii	2	16	4	18
δ Cephei	2	22	4	7
R Triang. Austr.	3	3	4	3
κ Pavonis	3	6	7	15
S Sagittae	3	10	6	20
RV Scorpii	3	12	4	21
V Velorum	3	21	4	20
V Centauri	4	8	5	19
T Velorum	4	15	6	0
SU Cygni	4	18	6	2
ζ Geminorum	4	23	9	23
U Vulpeculae	5	0	7	3
β Lyrae	5	6	8	13
S Normae	5	8	9	17
S Muscae	5	9	8	20
W Sagittarii	5	13	8	13
U Sagittarii	5	13	8	12
S Triang. Austr.	5	16	7	18
η Aquilae	6	3	8	12
X Sagittarii	6	12	9	9
R Triang. Austr.	6	13	7	13
T Vulpeculae	6	15	8	0
W Geminorum	7	3	9	18
S Crucis	7	13	9	1
U Aquilae	7	13	9	17
T Crucis	8	3	10	4
V Velorum	8	6	9	5
X Cygni	8	6	13	9
δ Cephei	8	7	9	16
Y Sagittarii	8	10	10	12
SU Cygni	8	14	9	22
V Carinae	8	15	10	17

	Minimum.		Maximum.	
	d	h	d	h
T Velorum Aug.	9	6	Aug. 10	15
RV Scorpii	9	14	10	23
V Centauri	9	19	11	6
R Triang. Austr.	9	22	10	22
TX Cygni	9	23	15	2
VY Cygni	10	2	12	4
T Vulpeculae	11	1	12	10
W Virginis	11	15	19	9
β Lyrae	11	17	14	19
S Sagittae	11	20	15	6
S Triang. Austr.	12	0	14	2
S Crucis	12	6	13	18
U Sagittarii	12	7	15	6
κ Pavonis	12	9	16	18
V Velorum	12	15	13	14
SU Cygni	12	11	13	19
U Vulpeculae	12	23	15	2
W Sagittarii	13	3	16	3
R Triang. Austr.	13	7	14	7
η Aquilae	13	7	15	16
X Sagittarii	13	12	16	9
δ Cephei	13	16	15	1
T Velorum	13	22	15	7
Y Sagittarii	14	3	16	5
U Aquilae	14	13	16	17
W Geminorum	14	18	17	9
T Crucis	14	20	16	21
S Muscae	15	0	18	11
S Normae	15	2	19	11
ζ Geminorum	15	3	20	3
V Centauri	15	7	16	18
V Carinae	15	7	17	11
T Vulpeculae	15	11	16	20
RV Scorpii	15	15	17	0
SU Cygni	16	7	17	15
R Triang. Austr.	16	16	17	16
S Crucis	16	22	18	10
V Velorum	17	0	17	23
VY Cygni	17	22	20	0
β Lyrae	18	4	21	11
S Triang. Austr.	18	8	20	10
T Velorum	18	13	19	22
U Sagittarii	19	0	21	23
δ Cephei	19	1	20	10
Y Ophiuchi	19	4	25	9
T Vulpeculae	19	22	21	7
Y Sagittarii	19	23	22	1
R Triang. Austr.	20	2	21	2
SU Cygni	20	3	21	11
S Sagittae	20	5	23	15
η Aquilae	20	11	22	20
X Sagittarii	20	13	23	10
W Sagittarii	20	17	23	17
V Centauri	20	19	22	6
U Vulpeculae	20	22	23	1
V Velorum	21	9	22	8
κ Pavonis	21	11	25	20
T Crucis	21	14	23	15
U Aquilae	21	14	23	18
S Crucis	21	15	23	3
RV Scorpii	21	17	23	2
V Carinae	22	1	24	5

Variable Stars of Short Period not of the Algol Type.—(Continued.)

	Maximum.		Minimum.			Maximum.		Minimum.	
	d	h	d	h		d	h	d	h
W Geminorum	Aug. 22	14	Aug. 25	5	S U Cygni	Aug. 27	20	Aug. 29	5
T Velorum	23	5	24	14	T Velorum	27	20	29	5
R Triang. Austr.	23	11	24	11	T Crucis	28	7	30	8
SU Cygni	23	23	25	7	W Sagittarii	28	8	31	8
T Vulpeculae	24	8	25	17	R V Scorpii	28	11	29	20
δ Cephei	24	10	25	19	U Aquilae	28	14	30	18
X Cygni	24	15	29	18	S Sagittae	28	14	32	0
β Lyrae	24	15	27	17	V Carinae	28	17	30	21
S Triang. Austr.	24	16	26	18	T Vulpeculae	28	19	30	4
S Muscae	24	16	28	3	U Vulpeculae	28	22	31	1
T X Cygni	24	17	29	20	W Virginis	28	22	37	14
S Normae	24	20	29	5	δ Cephei	29	18	31	3
ζ Geminorum	25	6	30	6	V Velorum	30	3	31	2
V Velorum	25	18	26	17	R Triang. Austr.	30	6	31	6
Y Sagittarii	25	18	27	20	W Geminorum	30	8	32	23
V Y Cygni	25	19	27	21	κ Pavonis	30	13	34	22
U Sagittarii	26	0	28	23	S Triang. Austr.	30	23	33	1
V Centauri	26	7	27	18	S Crucis	31	0	32	12
S Crucis	26	7	27	19	β Lyrae	31	2	34	9
R Triang. Austr.	26	21	27	21	Y Sagittarii	31	12	33	14
X Sagittarii	27	13	30	10	S U Cygni	31	16	33	0
η Aquilae	27	15	30	0	V Centauri	31	19	33	6

Maxima of RZ Lyræ.

Period $12^h\ 16^m\ 15^s.0$.

	d	h		d	h		d	h		d	h
July	1	9	July	16	17	Aug.	1	1	Aug.	16	9
	2	10		17	18		2	2		17	10
	3	10		18	19		3	2		18	11
	4	11		19	19		4	3		19	11
	5	12		20	20		5	4		20	12
	6	12		21	20		6	4		21	12
	7	13		22	21		7	5		22	13
	8	13		23	21		8	5		24	13
	9	14		24	22		9	6		25	14
	10	14		25	22		10	6		26	14
	11	15		26	23		11	7		27	15
	12	15		27	23		12	7		28	15
	13	16		29	0		13	8		29	16
	14	16		30	0		14	8		30	16
	15	17		31	1		15	9		31	17

Maxima of UY Cygni.

Period $13^h\ 27^m\ 27^s.59$. The minimum occurs $1^h\ 55^m$ before the maximum.

	d	h		d	h		d	h		d	h
July	1	14	July	17	6	Aug.	1	23	Aug.	17	16
	2	17		18	9		3	2		18	19
	3	20		19	12		4	5		19	22
	4	23		20	15		5	8		21	1
	6	1		21	18		6	11		22	4
	7	4		22	21		7	14		23	7
	8	7		24	0		8	17		24	10
	9	10		25	3		9	20		25	13
	10	13		26	6		10	23		26	16
	11	16		27	9		12	2		27	18
	12	19		28	12		13	5		28	21
	13	22		29	15		14	7		30	0
	15	0		30	18		15	10		31	3
	16	3		31	20		16	13			

Minima of Variable Stars of the Algol Type.

[Given to the nearest hour in Greenwich Mean Time. To reduce to Central Standard time subtract 6 hours, or for Eastern time subtract 5 hours, etc. Alternate minima only are given this month because of lack of space.]

U Cephei.

	d	h
July	4	17
	9	17
	14	16
	19	16
	24	15
	29	15
Aug.	3	15
	8	14
	13	14
	18	14
	23	13
	28	13

Z Persei.

	d	h
July	3	14
	9	17
	15	20
	21	23
	28	1
Aug.	3	4
	9	7
	15	10
	21	12
	27	15

Algol.

	d	h
July	1	15
	7	8
	13	2
	18	20
	24	13
	30	7
Aug.	5	0
	10	18
	16	12
	22	5
	27	23

RT Persei.

	d	h
July	1	8
	3	1
	4	18
	6	11
	8	3
	9	20
	11	13
	13	6
	14	23
	16	15
	18	8
	20	1
	21	18
	23	10
	25	3
	26	20
	28	13
	30	5
	31	22

RT Persei.

	d	h
Aug.	2	15
	4	8
	6	1
	7	17
	9	10
	11	3
	12	20
	14	12
	16	5
	17	22
	19	15
	21	8
	23	0
	24	17
	26	10
	28	3
	29	19
	31	12

λ Tauri.

	d	h
July	1	2
	9	0
	16	21
	24	19
Aug.	1	17
	9	15
	17	12
	25	10

Z Draconis.

	d	h
July	1	12
	4	6
	6	23
	9	16
	12	9
	15	2
	17	19
	20	12
	23	5
	25	23
	28	16
	31	9
Aug.	3	2
	5	19
	8	13
	11	6
	13	23
	16	16
	19	9
	22	2
	24	19
	27	13
	30	6

δ Libræ.

	d	h
July	2	8
	6	23
	11	15

δ Libræ.

	d	h
July	16	7
	20	22
	25	14
	30	6
Aug.	3	22
	8	13
	13	5
	17	21
	22	12
	27	4
	31	20

U Coronæ.

	d	h
July	4	1
	10	22
	17	20
	24	18
	31	15
Aug.	7	13
	14	11
	21	9
	28	6

R Aræ.

	d	h
July	2	21
	11	18
	20	14
	29	10
Aug.	7	7
	16	3
	24	28

U Ophiuchi.

	d	h
July	1	17
	3	9
	5	21
	7	13
	9	6
	10	22
	12	14
	14	7
	15	23
	17	15
	19	7
	21	0
	22	16
	24	8
	26	0
	27	17
	29	9
	31	1
Aug.	1	17
	3	10
	5	2
	6	18
	8	10
	10	3

U Ophiuchi.

	d	h
Aug.	11	19
	13	11
	15	3
	16	20
	18	12
	20	4
	21	20
	23	13
	25	5
	26	21
	28	13
	30	6
	31	22

Z Herculis

	d	h
July	4	5
	8	5
	12	5
	16	4
	20	4
	24	4
	28	4
Aug.	1	4
	5	3
	9	3
	13	3
	17	3
	21	3
	25	3
	29	2

RS Sagittarii.

	d	h
July	1	3
	5	23
	10	19
	15	15
	20	10
	25	6
	30	2
Aug.	3	22
	8	18
	13	14
	18	10
	23	6
	28	2

V Serpentis.

	d	h
July	3	12
	10	10
	17	8
	24	5
	31	3
Aug.	7	1
	13	23
	20	21
	27	18

RX Herculis.

	d	h
July	1	1
	2	20
	4	14
	6	9
	8	4
	9	22
	11	17
	13	12
	15	6
	17	1
	18	20
	20	15
	22	9
	24	4
	25	23
	27	17
	29	12
	31	7
Aug.	2	1
	3	20
	5	15
	7	9
	9	4
	10	23
	12	17
	14	12
	16	7
	18	1
	19	20
	21	15
	23	10
	25	4
	26	23
	28	18
	30	12

RV Lyræ.

	d	h
July	2	9
	9	13
	16	18
	23	23
	31	4
Aug.	7	8
	14	13
	21	18
	28	23

U Sagittæ.

	d	h
July	2	1
	8	19
	15	13
	22	7
	29	2
Aug.	4	20
	11	14
	18	8
	25	3
	31	21

Minima of Variable Stars of the Algol Type.—Continued.

SY Cygni.		SW Cygni.		UW Cygni.		Y Cygni.		VV Cygni.	
d	h	d	h	d	h	d	h	d	h
July 2	4	July 5	8	Aug. 6	1	July 23	14	July 13	4
14	4	14	12	12	23	26	14	16	3
26	5	23	15	19	20	29	14	19	2
Aug. 7	5	Aug. 1	19	26	18	Aug. 1	14	22	0
19	5	10	22	W Delphini.		4	14	24	23
31	6	20	2	July 2	18	7	13	27	22
		29	5	12	9	10	13	30	21
WW Cygni.		VW Cygni.		22	0	13	13	Aug. 2	20
July 4	2	4	15	31	15	16	13	5	19
10	18	21	12	Aug. 10	5	19	13	8	18
17	9	Aug. 7	9	19	20	22	13	11	17
24	0	24	6	29	11	25	13	14	16
30	15	UW Cygni.		Y Cygni.		28	13	17	14
Aug. 6	6	July 2	13	July 2	15	31	13	20	13
12	22	9	11	5	14	VV Cygni.		23	12
19	13	16	8	8	14	July 1	8	26	11
26	4	23	6	11	14	4	7	29	9
		30	4	14	14	7	6	UZ Cygni.	
				17	14	10	5	July 31	11
				20	14			Aug. 31	18

Approximate Magnitudes of Variable Stars on May 10, 1905.

[Communicated by the Director of Harvard College Observatory, Cambridge, Mass.]

Name.	R. A. 1900. h	m	Decl. 1900. °	′	Magn.	Name.	R. A. 1900. h	m	Decl. 1900. °	′	Magn.
T Androm.	0	17.2	+26	26	13 *d*	S Urs. Maj.	12	39.6	+61	38	12.5*d*
T Cassiop.	0	17.8	+55	14	8.5*d*	U Virginis	12	46.0	+ 6	6	14 *f*
R Androm.	0	18.8	+38	1	6.5	V "	13	22.6	— 2	39	8.5 *i*
S Ceti	0	19.0	— 9	53	*s*	R Hydrae	13	24.2	—22	46	4.4
S Cassiop.	1	12.3	+72	5	9.2*d*	S Virginis	13	27.8	— 6	41	12.5*d*
R Piscium	1	25.5	+ 2	22	*s*	R Can. Ven.	13	44.6	+40	2	8.3 *i*
U Persei	1	52.9	+54	20	10.5 *i*	S Bootis	14	19.5	+54	16	9.0 *i*
R Arietis	2	10.4	+24	36	*s*	R Camelop.	14	25.1	+84	17	8.4
o Ceti	2	14.3	— 3	26	*s*	R Bootis	14	32.8	+27	10	10.0 *i*
S Persei	2	15.7	+58	8	8.7 *i*	S Librae	15	15.6	—20	2	*u*
R Ceti	2	20.9	— 0	38	*s*	S Serpentis	15	17.0	+14	40	11.5 *i*
U "	2	28.9	—13	35	*s*	S Coronae	15	17.3	+31	44	9.2*d*
R Trianguli	2	31.0	+33	50	6.5 *i*	S Urs. Min.	15	33.4	+78	58	11
R Persei	3	23.7	+35	20	*f*	R Coronae	15	44.4	+28	28	12.5*d*
R Tauri	4	22.8	+ 9	56	*s*	V "	15	45.9	+39	52	9.0*d*
S "	4	23.7	+ 9	44	*s*	R Serpentis	15	46.1	+15	26	13 *d*
R Aurigæ	5	9.2	+53	28	9.1*d*	R Herculis	16	1.7	+18	38	15 *d*
U Orionis	5	49.9	+20	10	5.6 *i*	R Scorpii	16	11.7	—22	42	12 *i*
R Lyncis	6	53.0	+55	28	14 *f*	S "	16	11.7	—22	39	12 *d*
R Gemin.	7	1.3	+22	52	14 *f*	U Herculis	16	21.4	+19	7	8.2*d*
S Canis Min.	7	27.3	+ 8	32	11.0*d*	W Herculis	16	31.7	+38	32	*f*
R Cancri	8	11.0	+12	2	11.0	R Draconis	16	32.4	+66	58	11.5 *i*
V "	8	16.0	+17	36	12 0*d*	S Herculis	16	47.4	+15	7	8.5 *i*
S Hydrae	8	48.4	+ 3	27	8.0 *i*	R Ophiuchi	17	2.0	—15	58	*f*
T "	8	50.8	— 8	46	13 *f*	T Herculis	18	5.3	+31	0	10.0*d*
R Leo. Min.	9	39.6	+34	58	12.5*d*	R Scuti	18	42.2	— 5	49	5.5
R Leonis	9	42.2	+11	54	6.0 *i*	R Aquilae	19	1.6	+ 8	5	7.5*d*
R Urs. Maj.	10	37.6	+69	18	7.5	R Sagittarii	19	10.8	—19	59	*f*
R Comae Ber.	11	59.1	+19	20	15 *f*	S "	19	13.6	—19	12	*u*
T Virginis	12	9.5	— 5	29	12.5*d*	R Cygni	19	14.1	+49	58	12.3*d*
R Corvi	12	14.4	—18	42	12.1*d*	RT "	19	40.8	+48	32	9.5 *i*
Y Virginis	12	28.7	— 3	52	9.3 *i*	X "	19	46.7	+32	40	10.0*d*
T Urs. Maj.	12	31.8	+60	2	9.7*d*	S "	20	3.4	+57	42	11.5
R Virginis	12	33.4	+ 7	32	8.5*d*	RS "	20	9.8	+38	28	9.0*d*

Approximate Magnitudes of Variable Stars.—Continued.

Name.	R. A. 1900. h	m	Decl. 1900. °	′	Magn.	Name.	R.A. 1900. h	m	Decl. 1900. °	′	Magn.
R Delphini	20	10.1	+ 8	47	8.5*d*	S Lacertae	22	24.6	+39	48	*u*
U Cygni	20	16.5	+47	35	9.5 *i*	R "	22	38.8	+41	51	*u*
V "	20	38.1	+47	47	*f*	S Aquarii	22	51.8	−20	53	*s*
T Aquarii	20	44.7	− 5	31	*u*	R Pegasi	23	1.6	+10	0	*s*
R Vulpec.	20	59.9	+23	26	*u*	S "	23	15.5	+ 8	22	*s*
T Cephei	21	8.2	+68	5	6.3	R Aquarii	23	38.6	+15	50	*s*
S "	21	36.5	+78	10	9.2 *i*	R Cassiop.	23	53.3	+50	50	6.8 *i*

NOTE:—*f* denotes that the variable is probably fainter than the magnitude 13; *i*, that the light is increasing; *d*, that the light is decreasing; *s*, that it is near the Sun; and *u*, that its magnitude is unknown.

From observations made at the McCormick, Mt. Holyoke College and Harvard Observatories.

Maxima of Y Lyræ.

Period $12^h\ 03^m.9$. The minimum occurs $1^h\ 40^m$ before the maximum.

	d	h		d	h		d	h		d	h
July	1- 2	12	July	17-24	15	Aug.	1- 8	17	Aug.	17-24	19
	3- 9	13		25-31	16		9-16	18		25-31	20
	10-16	14									

GENERAL NOTES.

It is a long time since we have printed any strong mathematical articles, but this time, we give three such which are written by able astronomers. Our popular readers will indulge us, in this way, part of the time, for there are many other readers who like the relish of stronger meat now and then.

Dinner to Professor Young.—The Faculty of Princeton University, on May 17, gave Professor Charles A. Young an unusual mark of esteem, in tendering him a dinner at the Princeton Inn, as he retires from active service in the University. A number of distinguished gentleman were present and made after dinner speeches. The toast to which Professor Young fittingly responded, was:—

"He through heaven
That opened wide its blazing portals, led
To God's eternal house direct the way,
A broad and ample road, whose dust is gold,
And pavement stars."

Professor Young was given a large and handsome silver loving cup, of very graceful design, bearing on its face, in Greek, the following ascribed to Ptolemy:

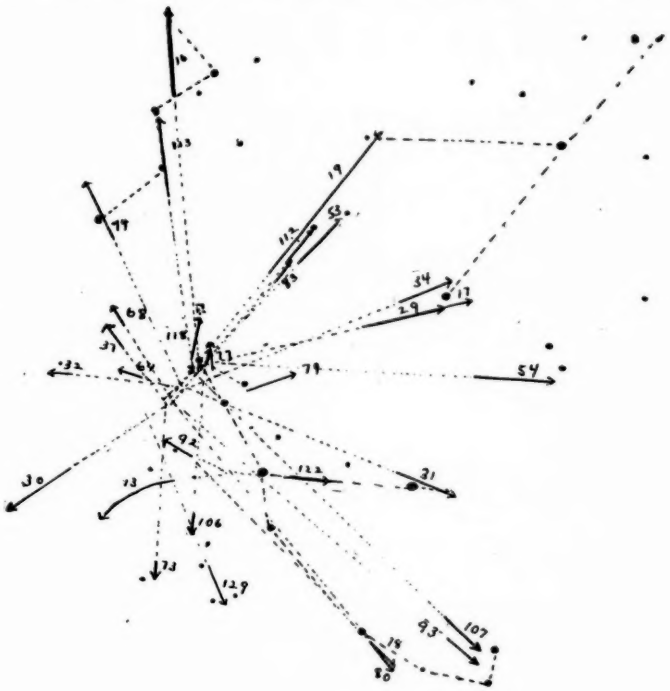
THE ASTRONOMER.

"I know that I am mortal and the creature of a day. Yet when I search the close-set whirling circles of the stars, no longer do I stand with feet upon the Earth, but seated with love himself I take my fill of the bread of Heaven."

Professor Young's active service, as Professor of Astronomy in Princeton University has lasted for twenty-eight years.

Washington Midnight Time Signal May 3, 1905.—At the request of the American Railway Association a midnight telegraphic time signal was sent out from Washington, May 3, 1905, in honor of the International Railway Congress which met at the Capital of the United States on the following day. A storm at the time prevented Goodsell Observatory from coöperating as requested.

Perseids, August 1904.—The weather in Arizona was not very favorable during August for meteor observations. There were continual thunder showers during August, which made the sky overcast every night. Some evenings were clear for a short while; the 12th of August, was clear for several hours, although lightning flashed brilliantly all night long, and prevented a very complete watch of the sky. 143 Perseids were observed in all, and a few were plotted on a map. There were a number of meteors close to the radiant, so that an accurate radiant point could be determined. The meteors were rather few and fainter than usual, but the small number may be partly due to lightning.



PERSEIDS, AUGUST, 1904.

The 11th and 13th of the month were completely cloudy, so that it was impossible to tell when the maximum was reached. Besides Perseids there were a number of shooting stars from Cygnus and Cassiopeiæ. One beautiful shooting star rivalling Venus in brilliancy burst over the mountains in the northwest, plainly showing the outlines of the peaks.

No.	Date.	Magn.	Color.	Time.	Remarks.
1.	Aug. 6	3	W		5th, W. trail
2.		5	W		2 shooting stars
3.		5	W		
4.	9	1	W		4th, W. trail
5.		4	W		2 shooting stars

No.	Date.	Magn.	Color.	Time.	Remarks.
6.	Aug. 10	4	W		6th, W. trail
7.		3	W		5th, W. trail
8.		3	W		3 shooting stars
9.		3	W		5th, W. trail
10.		2	W		4th, W. trail
11.		3	B		
12.		2	W		
13.		3	W		
14.		6	W		11 shooting stars
15.		6	W		
16.		0	Or.		4th, W. trail
17.		2	W		
18.		2	W		
19.		0	Or.		4th, W. trail
20.	12	0	Or.	10:04	4th, W. trail
21.		5	W	10:09	
22.		5	W	10:13	
23.		4	W	10:20	
24.		2	W	10:36	
25.		2	B	10:40	4th, W. trail
26.		4	W	10:40	
27.		0	W	11:21	5th, W. trail
28.		1	W	11:26	9 shooting stars
29.		2	W	11:28	4th, B. trail
30.		4	W	11:35	
31.		3	W	11:38	
32.		3	W	11:45	
33.		3	W	11:47	
34.		6	W	11:50	
35.		4	W	11:52	
36.		3	W	11:57	9 shooting stars
37.		4	W	12:00	
38.		3	W	12:00	
39.		5	W	12:04	4 shooting stars
40.		3	W	12:06	—4 shooting stars, burst
41.		1	W	12:08	
42.		4	W	12:10	
43.		2	W	12:11	
44.		3	W	12:12	
45.		1	W	12.13	
46.		2	W	12:14	
47.		2	W	12:14	5th, W. trail
48.		1	R	12:25	5th, W. trail
49.		2	W	12:27	
50.		0	W	12:28	
51.		3	W	12:40	
52.		0	W	12:43	
53.		3	B	12:44	5th, W. trail
54.		3	W	12:47	
55.		0	W	12:48	
56.		1	W	12:50	
57.		0	W	12:52	5th, W. trail
58.		5	B	12:53	
59.		1	W	12:53	5th, W. trail
60.		5	W	12:55	12 shooting stars
61.		—1	W	12:58	5th, W. trail
62.		3	W	12:59	
63.		3	W	1:04	5th, W. trail
64.		5	W	1:05	
65.		0	W	1:07	
66.		1	W	1:10	
67.		—1	W	1:12	
68.		5	W	1:14	

No.	Date.	Magn.	Color.	Time.	Remarks.
69.	Aug. 12	4	W	1:16	
70.		2	W	1:20	7 shooting stars
71.		0	W	1:21	5th, W. trail
72.		1	W	1:26	4th, W. trial
73.		4	W	1:27	5th, W. trail
74.		3	W	1:30	
75.		0	W	1:31	5th, W. trail
76.		5	B	1:32	
77.		5	B	1:34	
78.		3	W	1:34	
79.		3	W	1:34	
80.		3	B	1:35	5th, W. trail
81.		0	W	1:37	
82.		2	B	1:39	
83.		5	B	1:41	
84.		3	R	1:51	
85.		4	W	1:55	
86.		4	W	1:56	
87.		1	W	1:57	
88.		4	W	1:59	
89.		4	W	2:01	
90.		4	W	2:02	5th, W. trail
91.		0	R	2:02	
92.		—	—	2:06	15 shooting stars
93.		6	W	2:07	
94.		0	W	2:10	
95.		0	W	2:14	5th, W. trail
96.		3	W	2:18	
97.		3	W	2:20	
98.		0	W	2:28	5th, W. trail
99.		0	W	2:30	5th, W. trail
100.		4	W	2:31	
101.		1	R	2:32	4 shooting stars
102.		0	W	2:35	5th, W trail
103.		6	W	2:37	
104.		6	W	2:39	
105.		0	W	2:39	5th, W. trail
106.		3	W	2:42	Cloudy
107.		6	B	2:44	
108.		3	W	2:50	
109.		5	W	2:56	
110.		0	W	3:02	
111.		−3	W	3:04	4th, W. trail
112.		4	W	3:06	
113.		2	W	3:12	
114.		6	W	3:17	
115.		4	B	3:18	
116.		0	W	3:20	5th, W. trail
117.		0	W	3:21	
118.		3	W	3:26	
119.		1	W	3:26	
120.		2	W	3:28	
121.		2	W	3:32	5th, W. trail
122.		3	W	3:41	
123.		3	W	3:44	
124.		3	W	3:58	
125.		1	W	3:58	5th, W. trail
126.		4	W	4:02	
127.		2	W	4:04	17 shooting stars
128.		1	W	4:06	5th, W. trail
129.		5	W	4:09	
130.		5	W	4:12	
131.		4	W	4:18	7 shooting stars

No.	Date.	Magn.	Color.	Time.	Remarks.
132.	Aug. 12	1	W	4:21	
133.		5	W	4:24	
134.		5	W	4:25	
135.		5	W	4:27	
136.		4	W	4:28	
137.		3	W	4:31	
138.		4	W	4:33	
139.		2	W	4:35	
140.		3	W	4:38	
141.		3	W	4:40	
142.		0	W	4:45	
143.		3	W	4:45	

Total, 143 Perseids; 100 shooting stars. 123 Perseids in 6 hours 41 minutes or an average of one in every 3¼ minutes. Numbers 13, 16, 17, 19, 29, 30, 31, 32, 34, 37, 54, 55, 64, 68, 73, 74, 77, 78, 79, 80, 83, 93, 106, 107, 112, 113, 118, 122 and 128 are plotted on the map. ROBERT M. DOLE.

FLAGSTAFF, Arizona.

Eros Photographed at Arequipa.—A letter from the Arequipa Station of this observatory states that Eros has been photographed there by Mr. Manson with the Bruce Telescope, in the positions given below.

Plate A.	Date. 1905.	G. M. T. Exposure. h m	m	R.A. 1900. h m s	Dec. 1900. ° ′
7255	April 11	19 57	70	20 36 37	−25 04.5
7256	April 12	20 41	134	20 38 34	−24 55.6
7257	April 14	20 40	45	20 42 12	−24 39.1

These positions agree closely with those given in Circular No. 73, and appear to be the first obtained since its recent conjunction with the Sun.

NOTE:—The date of Neostyle No. 192 should be May 27, 1905.

HARVARD OBSERVATORY. EDWARD C. PICKERING.

The Lawrence Fellowship.—Mr. Percival Lowell has again manifested his interest in the science of Astronomy by establishing a fellowship—to be known as the Lawrence Fellowship—for the department of Astronomy at Indiana University. This fellowship is annually available and will cover the college calendar year, that is from Commencement to Commencement. The Lawrence Fellows will be appointed by the department, but the donor reserves the right of final passing upon the suitability of the candidate so presented. He shall be given an opportunity to do astronomical research at Lowell Observatory and to prepare a thesis upon some astronomical subject acceptable to the Fellow and the Director of the Observatory. The University of Indiana will grant to him the degree of Master of Arts upon the presentation of this thesis, and the fulfillment of certain other university requirements.

The Fellow will receive $600 and his travelling expenses to and from the Observatory and will give general assistance in the work of the observatory during the period of his Fellowship.

So liberal an endowment will promote the interests of this science by attracting to it students of ability and by enabling young men with zeal for astronomical research to add something to human knowledge.

The Lawrence Fellow for 1905-06 is John Charles Duncan who graduates from Indiana University with the Class of 1905.

A Road-Book to the Stars.—It may be remembered by our readers who see *Knowledge* that T. E. Heath of 53, Park Place, Cardiff, has recently written some articles for that publication concerning the parallaxes and the magnitudes

of the heavenly bodies. His lively interest in this and kindred themes belonging to Astronomy has led to the publication of a book for popular readers with title, "Our Stellar Universe, A Road-Book to the Stars."

The leading thought in this new book is: By a simple scale, fortunately noticed, the distance of the stars and their glory is made manifest. The author has made six drawings by which, starting from the Earth, he conducts his readers to the supposed boundary of the Stellar Universe. On the way he has made two of the pictures to be used with the stereoscope. These pictures include the stars whose parallaxes are known approximately. Those who have used the stereoscope know that the effect on the plain picture rightly prepared for that instrument is to make the stars appear as if hung in space with clear and surprising relief. The idea is not a new one in astronomy but to prepare such pictures and have them bound in a book, with perforated leaves, so that they may be detached and used in a common stereoscope, is new to us. At Goodsell Observatory pictures of the Moon have been taken when full and "running high" and when full and "running low" and placed side by side on a card. Such an arrangement in the stereoscope makes the Moon appear like a sphere in space, with so much naturalness and certainty that the view is very pleasing.

Another feature in the first chapter of the text that is new, is the unit of distance chosen by the author to make the measures of the star depths possible to comprehend. He does it by using the distance that light travels in one year and representing that unit by one mile; then the distance of the Earth from the Sun will be represented by one inch on the same scale. The error in English measure is about 0.0005 of an inch, which is pretty large in a celestial scale that will still employ large numbers. For his purpose in an elemental popular presentation of his theme it is, of course, near enough the known truth to be helpfully instructive. We wish that the author's full page etchings had been not quite so much crowded. The plan is good, but he has tried to show so much in small space that the eye does not easily follow out the details. The book is a small quarto in form, contains seventy-three pages, has a number of useful tables at the end of the text: costs five shillings net and it may be secured from "*Knowledge*" office, or from King, Shell & Olding; both at 27 Chancery Lane, London, W. C. The publishers have done their work neatly and well.

New Asteroids.—The following have been added to the list of new minor planets since our last note:—

		Discovered by	at		Local Mean Time.	R. A.	Decl.	Mag.
					h m	h m	° ′	
1905	QH	Wolf	Heidelberg	1905	April 3 12 53.4	13 50.6	+ 1 57	13.3
	QJ	Wolf	"		May 7 10 53.7	14 55.8	−22 55	11.7
	QK	Götz	"		April 6 12 56.9	14 52.4	− 4 49	12.0
	QL	Wolf	"		May 7 10 53.7	15 09.1	−20 39	13.0
	QM	Götz	"		9 11 27.1	15 34.9	− 6 55	12.5
	QN	Wolf	"		9 13 28.7	16 28.4	−17 09	12.8

New Variable 60.1905 Ophiuchi.—This variable was discovered by Mr. Thomas D. Anderson, of Northrig, Haddington, Scotland. On March 28, 1905, it was of magnitude 9.8 but on May 4 it was only 11.0 The positions for 1855.0 of the variable and some neighboring stars are as follows:

	R. A.			Decl.		Mag.
	h	m	s	°	′	
60.1905 Ophiuchi	16	45	40	+ 5	38.5	——
BD. 5° 3285	16	45	28	+ 5	10.3	9.2
a	16	45	21	+ 5	47.1	10.1
b	16	45	34	+ 5	45.0	10.8
c	16	45	51	+ 5	31.7	10.2

Erratum.—Attention is called to the fact that the middle of the second paragraph of Mr. Pickering's article "Mars," on page 305, should read "bright enough for snow."

PUBLISHER'S NOTICES.

Contributors are asked to prepare copy carefully, and write *all proper names very plainly*. If other language than the English is used to any considerable extent it should be type-written. Manuscript to be returned should be accompanied by postage for that purpose.

All Drawings for publication should be done in India ink, twice the size that the cut will be on the printed page. The lines, figures and letters should be made even, very smooth and uniformly black in every part of the copy, in order to secure the best reproductions possible by the modern quick processes of engraving now most generally used.

Proofs will generally be sent to authors living in the United States, if copy is furnished before the first of the month preceding that of publication. We greatly prefer that authors should read their own proofs, and we will faithfully see that all corrections are made in the final proofs.

Renewals.—Notices of expiration of subscription are sent directly after the last number of this publication for which payment has been made. It is especially requested that subscribers will send renewal, or order for renewal, promptly as this publication will not be continued beyond the time for which it has been ordered.

Astronomical News.—It is very much desired on the part of the management, that brief news paragraphs of astronomical work in all the leading observatories of the United States be furnished to this publication, as regularly and as often as possible.

The work of amateur astronomers, and the mention of "personals" concerning prominent astronomers will be welcome at any time.

The building and equipping of new observatories, the manufacture, sale or purchase of new astronomical instruments, with special reference to improvements and new designs, and the results of new methods of work in popular language, will be deemed very important matter and will receive prompt attention. It is greatly desired that all persons interested bear us in mind and promptly respond to these requests.

Messrs. Wm. Wesley & Son, 28 Essex Street, Strand, London, England, are our sole European agents.

Reprints of articles for authors, when desired, will be furnished in titled paper covers at small cost. Persons wanting reprints should order them when the manuscript is sent to the publisher. They cannot be furnished later without incurring much greater expense.

Subscription Price to Popular Astronomy in the United States, Canada and Mexico is $2.50 per volume of 10 consecutive numbers. Price per volume of 10 numbers to foreign subscribers $3.00. Postage will be accepted only in payment of bills of fifty cents, or less. Volume begins with January number, but subscriptions may begin with any number.

All correspondence and all remittances should be sent to

WM. W. PAYNE,
Northfield, Minn., U. S. A.